YuMi Deadly Maths Program

Pre-Prep to Year 7: Resource 5

Algebra

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

http://ydc.qut.edu.au

Project funded by the Queensland Department of Education and Training

Queensland Government

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ACKNOWLEDGEMENTS

We acknowledge the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a Research Centre within the Faculty of Education at QUT which aims to improve the mathematics learning and the employment and life chances of Aboriginal and Torres Strait Islander and low SES students at early childhood, primary and secondary levels, in vocational education and training courses, and through a focus on community within school and neighbourhood.

“YuMi” is a Torres Strait Islander Creole word meaning “you and me” but is used here with permission from the Torres Strait Islanders’ Regional Educational Council to mean working together as a community for the betterment of education for all. “Deadly” is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre’s motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates YDC’s focus which is encapsulated by the slogan: Growing community through education.

The YuMi Deadly Centre can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

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THE YUMI DEADLY MATHS PROGRAM

Since 2000, researchers now in the YuMi Deadly Centre have been collaborating with principals and teachers predominantly from Aboriginal and Torres Strait Islander schools and occasionally from low socio-economic status (SES) schools in a series of small projects to enhance student learning of mathematics. Most commonly these small projects have focused on a particular mathematics strand (e.g., whole-number numeration, operations, algebra, measurement) or on a particular part of schooling (e.g., middle school teachers, teacher aides, parents). They have resulted in the development of specialist materials but not a complete mathematics program (these specialist materials can be accessed via the YuMi Deadly Centre’s website, http://ydc.qut.edu.au).

However, in October 2009, the YuMi Deadly Centre received funding from the Queensland Department of Education and Training (DET) through the Indigenous Schooling Support Unit (ISSU) Central-Southern Queensland to develop a train-the-trainer project, called the Teaching Indigenous Mathematics Education or TIME project, to enhance the capacity of more than 40 schools with Aboriginal and Torres Strait Islander students to effectively teach mathematics to their students. In November 2009, the final decision of the DET TIME project was to focus on Central and Southern Queensland Indigenous and low SES communities. The project will run for three years, focusing on Years Pre-Prep to 3 in 2010, Years 4 to 7 in 2011 and Years 8 to 9 in 2012, to cover all mathematics strands: Number, Operations, Patterns & Algebra, Space, Measurement, and Chance & Data.

This program is called the YuMi Deadly Maths Program and is designed to enhance mathematics learning outcomes, improve participation in higher mathematics subjects, and improve employment and life chances and participation in tertiary courses. The Program is unique in its focus on creativity, structure and culture with regard to mathematics and on whole-of-school change with regard to implementation. The Centre believes that changing a mathematics program will not improve mathematics learning unless accompanied by a whole-of-school program to challenge attendance and behaviour, encourage pride and self belief, instil high expectations, and build local leadership and community involvement. The Program is strongly influenced by the philosophy of the Stronger Smarter Institute (Sarra, 2003) that any school has the potential to meet the challenges of successfully teaching their students. The underpinning philosophy of the Program is applicable to all schools with high numbers of students at risk, as the core of mathematics teaching, school change and leadership, and contextualisation and cultural empowerment ideas that are advocated by the YuMi Deadly Centre represent best practice for all students to ameliorate the effect of traditional Eurocentric schooling.

This resource is the fifth resource in a series of eight to be produced in 2011 that fully describe the YuMi Deadly Maths Program for Years Pre-Prep to 7. This resource focuses on teaching Algebra for Pre-Prep to Year 3 and covers patterns, arithmetic principles, equations and functions. It overviews the mathematics and provides a teaching framework for Algebra, and describes classroom activities for Years PP-3. It takes account of ACARA and is a resource in continuous development.
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Overview and Purpose</td>
<td>1</td>
</tr>
<tr>
<td>1.1. Sequencing algebra</td>
<td>1</td>
</tr>
<tr>
<td>1.2. Content framework for algebra</td>
<td>2</td>
</tr>
<tr>
<td>1.3. Teaching algebra</td>
<td>3</td>
</tr>
<tr>
<td>2. Repeating and Growing Patterns</td>
<td>5</td>
</tr>
<tr>
<td>2.1. Pattern types, sequences and materials</td>
<td>5</td>
</tr>
<tr>
<td>2.2. Very early patterning activities</td>
<td>8</td>
</tr>
<tr>
<td>2.3. Early to middle patterning activities</td>
<td>10</td>
</tr>
<tr>
<td>2.4. Later patterning activities</td>
<td>14</td>
</tr>
<tr>
<td>3. Arithmetic Principles</td>
<td>17</td>
</tr>
<tr>
<td>3.1. Equivalence, order and Field structure</td>
<td>17</td>
</tr>
<tr>
<td>3.2. Very early Arithmetic Principles activities</td>
<td>19</td>
</tr>
<tr>
<td>3.3. Early to middle Arithmetic Principles activities</td>
<td>22</td>
</tr>
<tr>
<td>3.4. Later arithmetic principles activities</td>
<td>27</td>
</tr>
<tr>
<td>4. Equivalence and Equations</td>
<td>29</td>
</tr>
<tr>
<td>4.1. Major ideas and models</td>
<td>30</td>
</tr>
<tr>
<td>4.2. Very early Equivalence and Equation activities</td>
<td>32</td>
</tr>
<tr>
<td>4.3. Early to middle Equivalence and Equation activities</td>
<td>37</td>
</tr>
<tr>
<td>4.4. Later equivalence and equation activities</td>
<td>42</td>
</tr>
<tr>
<td>5. Change and Functions</td>
<td>49</td>
</tr>
<tr>
<td>5.1. Major Ideas and models</td>
<td>49</td>
</tr>
<tr>
<td>5.2. Very early Change and Functions activities</td>
<td>53</td>
</tr>
<tr>
<td>5.3. Early to middle Change and Function activities</td>
<td>55</td>
</tr>
<tr>
<td>5.4. Later Change and Function activities</td>
<td>58</td>
</tr>
<tr>
<td>6. Cultural implications</td>
<td>63</td>
</tr>
<tr>
<td>6.1. Algebra as abstraction of abstraction</td>
<td>63</td>
</tr>
<tr>
<td>6.2. Abstraction, structure and holistic teaching</td>
<td>64</td>
</tr>
<tr>
<td>6.3. Indigenous culture, mathematics and holistic teaching</td>
<td>65</td>
</tr>
<tr>
<td>6.4. Cultural implications for teaching Algebra</td>
<td>67</td>
</tr>
<tr>
<td>7. Teaching Algebraic Structure</td>
<td>69</td>
</tr>
<tr>
<td>7.1. Connections and big ideas</td>
<td>69</td>
</tr>
<tr>
<td>7.2. The power of algebraic big ideas</td>
<td>71</td>
</tr>
<tr>
<td>7.3. RAMR Cycle</td>
<td>76</td>
</tr>
<tr>
<td>7.4. Sequences and models to scaffold abstraction</td>
<td>78</td>
</tr>
<tr>
<td>8. Teaching Framework for Algebra PP-7</td>
<td>81</td>
</tr>
<tr>
<td>8.1. Justifying the framework</td>
<td>81</td>
</tr>
<tr>
<td>8.2. Yearly teaching frameworks</td>
<td>85</td>
</tr>
<tr>
<td>References</td>
<td>89</td>
</tr>
</tbody>
</table>
List of Figures and Tables

Figure 1. Algebra sections and sequence .......................................................... 1
Figure 2. Reality-Abstraction-Mathematics-Reflection (RAMR) framework ......................... 3
Figure 3. Sequence for teaching many algebraic topics ............................................. 4
Figure 4. Teaching sequence for patterns ................................................................... 5
Figure 5. Sequence for teaching Arithmetic Principles ............................................. 17
Figure 6. Sequence for teaching Equivalence and Equations ..................................... 29
Figure 7. Sequence for teaching Change and Functions ........................................... 49
Figure 8. Algebra as an abstraction of an abstraction ............................................... 64
Figure 9. Connections between strands and within Number strand ............................. 70
Figure 10. Relationship between perceived reality and created mathematics .................... 76
Figure 11. RAMR framework ................................................................................ 77
Figure 12. Sequence of concrete material representations .......................................... 78
Figure 13. Discrete vs Continuous ........................................................................... 78
Figure 14. Sequence of working from concrete materials to abstract symbols .................. 79
Figure 15. The Payne and Rathmell (1977) triangle for early number ............................ 79

Table 1. The framework for teaching for Algebra ..................................................... 2
1. Overview and Purpose

Algebra is a new topic in the primary school years but a traditional topic in the secondary school years. Thus, in both schooling levels, primary and secondary, there is a need for change. The primary years are doing a new topic not part of traditional primary mathematics. The secondary years now have primary knowledge of algebra to follow and must realise that what they traditionally taught may have been covered or needs linking to what has been covered.

Algebra is the generalisation of arithmetic (number and operations) and, as such, is a second level abstraction (see Section 6.1). Arithmetic sentences like 2+3=5 are powerful generic descriptions of our world in that they describe 2 joining 3 in any context (e.g., shopping, fishing, sport). However, algebraic sentences like \( a+b=b+a \) take this further and are even more powerful and generic descriptions of any number in any context in our world, and later any construction (e.g., flight of a plane) in reality that can be represented algebraically.

1.1. Sequencing algebra

In the early years, algebra is not about x’s and y’s; it is about doing and understanding arithmetic in a deeper way that builds arithmetic structure and prepares students for algebra. In the later upper primary and early secondary years, it is about understanding the world algebraically, manipulating equations and expressions, solving equations, and expressing and representing functions.

Since Algebra is about generalisation it includes patterns as training in the act of generalisation. It also includes generalisations of arithmetic that hold for all numbers, that is, the principles such as inverse and commutativity. Finally, mathematics has two major approaches: (1) relationships between things that in arithmetic and algebra are represented predominantly by equations; and, (2) change or transformation from one thing to another that is represented by Arrowmath notation and leads to functions. Thus, in this booklet, algebra is built around the four sections and the sequences outlined as in Figure 1 on the right. More detailed sequences are given in Sections 2 to 5.

Figure 1. Algebra sections and sequence
1.2. Content framework for algebra

The content framework (Table 1) shows the organisation of algebra into a framework of four topics with each of these topics partitioned into sub-topics. These define most of the terms used in Figure 1. They relate to Sections 2 to 5 where more detail will be given as well as ideas for teaching the components: Section 2 is on patterns, Section 3 on Arithmetic principles, Section 4 on Equations and Section 5 on Functions.

Table 1. The framework for teaching for Algebra

<table>
<thead>
<tr>
<th>TOPICS</th>
<th>SUB-TOPICS</th>
<th>DESCRIPTIONS AND CONCEPTS/STRATEGIES/WAYS</th>
</tr>
</thead>
<tbody>
<tr>
<td>Repeating and growing patterns</td>
<td>Repeating patterns</td>
<td>Following/copying patterns; Continuing patterns; Completing patterns; Constructing patterns; Identifying repeats</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relating position to item; Finding rule for this relationship; Numbers → language → variable.</td>
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<tr>
<td></td>
<td></td>
<td>Generalising repeat; Extending to equivalent fractions and proportion</td>
</tr>
<tr>
<td></td>
<td>Growing patterns</td>
<td>Copying, continuing, completing &amp; creating patterns, Objects → numbers; Visual analysis → table analysis.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relating position to term (visual → table); Finding pattern rule (sequential and position - numbers → language → variables); One, two or more operations (+/- → x/÷).</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalising sequencing &amp; pattern rules Relating pattern rule, pattern and graph. Introducing variable and algebraic expressions</td>
</tr>
<tr>
<td></td>
<td>Patterns in other strands</td>
<td>Using patterns to find number and operation ideas and relationships.</td>
</tr>
<tr>
<td>Arithmetic principles</td>
<td>Equals and order</td>
<td>Reflexivity, symmetry, transitivity, well ordered</td>
</tr>
<tr>
<td></td>
<td>Number size</td>
<td>How other numbers changes in operations, e.g., compensation, inverse relation, equivalence</td>
</tr>
<tr>
<td></td>
<td>Field</td>
<td>Identity, inverse, commutativity, associativity, distributivity,</td>
</tr>
<tr>
<td></td>
<td>Manipulation of algebraic expressions</td>
<td>Substitution, simplification, operations with algebraic expressions and factorisation</td>
</tr>
<tr>
<td>Equations and inequations</td>
<td>Meanings of equals and equals/order principles</td>
<td>Same-different/equals-unequals; Mass and length models; Unnumbered to numbered contexts</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Relation real world stories to equivalence &amp; equations</td>
</tr>
<tr>
<td></td>
<td>Balance rule</td>
<td>Un-numbered → materials → generalised;</td>
</tr>
<tr>
<td></td>
<td>Unknowns/variable</td>
<td>Representing unknowns and unknowns/variables in equations</td>
</tr>
<tr>
<td></td>
<td>Solutions</td>
<td>Solving for unknowns in equations</td>
</tr>
<tr>
<td>Change and functions</td>
<td>Meanings and notation</td>
<td>Change; Function machine; Input-output tables; Unnumbered → numbered; Relating stories to change</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Arrowmath notation; Arithmetic excursions; Relation to equations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Generalisation of change rules (numbers → language → variables)</td>
</tr>
<tr>
<td></td>
<td>Backtracking</td>
<td>Inverse; Backtracking; One, two and more changes; One or more operations (+/- → x/÷)</td>
</tr>
<tr>
<td></td>
<td>Solutions</td>
<td>Consider actions of unknowns; Using backtracking to solve equations</td>
</tr>
<tr>
<td></td>
<td>Graphing</td>
<td>Graphing change; Relating graphs, equations, Arrowmath notation, input-output and real world stories</td>
</tr>
</tbody>
</table>
1.3. Teaching algebra

General teaching across the program

This booklet, and the teaching of algebra that is described in it, is part of the YuMi Deadly Maths (YDM) program. This program has four components as follows.

1. Culture, community & whole school – YDM programs take account of culture, involve community and are based on whole school programs.

Interestingly, although it is commonly seen as an extension of arithmetic, and thus seen as more difficult than arithmetic, Algebra may be an excellent way to teach mathematics to Aboriginal, Torres Strait Islander and low SES students. This is because understanding is based on seeing mathematics as a whole structure and this appears to reflect learning styles for these students. This is argued in more detail in Section 6.

2. RAMR cycle – YDM teaching is based on a cycle of teaching built around four stages, namely, reality, abstraction, mathematics, and reflection. This is diagrammatically illustrated in Figure 2 below. More detail is described in Section 7.

![Reality-Abstraction-Mathematics-Reflection (RAMR) framework](image)

Figure 2. Reality-Abstraction-Mathematics-Reflection (RAMR) framework

3. Connections and big ideas – YDM sees mathematics as a structure, a language and a tool for problem solving and reasoning.

Because it is a generalisation of arithmetic, algebra is based on the big ideas and connections of arithmetic. As the best way to learn mathematics is by using these connections and big ideas, algebra is important for arithmetic as arithmetic is important for algebra. This is why algebra has been brought into the primary school – to enable better understanding of number and operations and better preparation for formal algebra. Fortunately, algebra has a lot of interesting activities that assist necessary number and operations work to be learnt but also pre-empt later algebra work. This is argued in more detail in section 6.
4. **Teaching framework** – it provides a framework of content (i.e., topics and sequences) and proficiencies (i.e., understanding and fluency). This is based on ACARA but adds richer information to show how big ideas and connections link into the year levels and the ACARA content statements.

**Specific teaching of algebra**

Algebra instruction is directed towards generalities. A method that appears to be effective is to start instruction with un-numbered situations. This is because students appear to more easily look for patterns in un-numbered activities than in numbered situations where they tend to look for answers. Thus, the activities in this booklet, as far as possible, start with un-numbered activities, then move to numbered activities and then to variable activities. This involves generalisation in which students progress from working with small numbers, to large numbers (this is called quasi-generalisation), everyday language, and finally, formally with variables. Figure 3 illustrates this diagrammatically.

![Diagram](image)

*Figure 3. Sequence for teaching many algebraic topics*

[Note: This booklet should be read in relation to the first two YDM Resources – *Philosophy and Pedagogy*, and *School Change and Leadership.*]
2. Repeating and Growing Patterns

There are two pattern types to explore in Algebra to promote early algebraic thinking, namely repeating and growing patterns. Repeating patterns are simplest for introducing activities that engage students in noticing and identifying patterns, whereas growing patterns introduce more complex relationships between terms. Here we begin with repeating patterns. As well, patterns in other strands of mathematics can be identified and lead to deeper understandings of those strands – this is particularly in terms of number and operations. A sequence for repeating and growing patterns is in Figure 4.

![Figure 4. Teaching sequence for patterns]

2.1 Pattern types, sequences and materials

Repeating patterns

Repeating patterns are linear sequences of objects, pictures or numbers that form a pattern because a section of them repeats; for example:

\[ 0 \times 0 \times 0 \times 0 \times 0 \times 0 \]  repeating part: 0 x

\[ o l l o l l o l l o l l o l l o l l o l l o l \]  repeating part: o l l

The crucial thing is to be able to go from pattern to repeating part and repeating part to pattern. The normal sequence of activities is as follows:

(a) copying, continuing and completing repeating patterns and identifying repeating part;
(b) constructing repeating patterns (without direction and when given a repeating part);
(c) finding what object is at a position or finding positions for objects (e.g., what object is the 13th term?; what terms are square red counters?)
(d) breaking pattern into repeats and connecting to growing patterns, e.g.,

\[ 0 \times 0 \times 0 \times 0 \times \ldots \rightarrow 0x\ 0x\ 0x\ \ldots \rightarrow 0x\ 0xx\ 0xx\ 0xx\ 0xx\ \ldots \];
(e) representing repeats on tables and generalising tabled numbers to variables; and
(f) using tables of repeats to introduce fractions, equivalent fractions, ratio and equivalent ratio (proportion).
The common materials to be used consist of objects of different colour, size, shape, etc., with any one or more attributes determining the basis of the pattern, plus small numbered cards (1 to 5), tables, and calculators. It is useful to have magnetic copies of these objects/cards that can be placed on whiteboards.

The major ideas to be developed from repeating patterns are: (a) generalisation, (b) representing generalisations with variables (introduction to algebra), (c) fractions and ratio, and (d) equivalent fraction and ratio.

**Growing patterns**

Growing patterns are series of terms where there is a fixed part and a growing part on right. In the pattern on the right, 0 is fixed and X is growing.

When the growing part does not grow (grows by zero), you have a repeating pattern as on right.

It is possible for the fixed part to not exist (to be zero) as on right.

The focus on growing patterns is to identify what is called the pattern rule which describes the growth. For patterns like that on the right, there are two types of rules.

**Sequential**:
The nth term is the previous term +1

**Position**:
The nth term is 1 + n, since:

<table>
<thead>
<tr>
<th>Term</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st term</td>
<td>1 O and 1 X (1 + 1)</td>
</tr>
<tr>
<td>2nd term</td>
<td>1 O and 2 Xs (1 + 2)</td>
</tr>
<tr>
<td>3rd term</td>
<td>1 O and 3 Xs (1 + 3)</td>
</tr>
<tr>
<td>and so on</td>
<td></td>
</tr>
</tbody>
</table>

Position rules enable growing patterns to be used to introduce the notion of variable. They also can be used to plot graphs. When this is done, a relation exists between the graph, the growing part and the fixed part—the growing part is the slope and the fixed part is the y intercept (for $y = x+1$, slope is 1 and y intercept is 1) as is shown on right.

Note: In previous times, sequential pattern rules (e.g., “1 more”) were considered to be trivial. With the growth of computers this has changed. In the example above, the position pattern rule gives the function $y = x+1$ but the sequential pattern rule gives the function $y(1) = 2$, $y(k+1) = y(k)+1$. This is now how functions are represented in programming.

For all activities, the crucial thing is to go from pattern to pattern rule and pattern rule to pattern. The normal sequence of activities is as follows:

(a) copying, continuing and completing growing patterns;

(b) constructing growing patterns (without direction and when given a pattern rule);
(c) finding what objects are at a position and what position has certain objects (e.g., what term has the 20th red circle?);

(d) identifying growing and fixed parts of visual patterns;

(e) identifying pattern rules (sequential and position) from visual patterns with and without use of number tables;

(f) identifying growing and fixed parts and pattern rule from number tables;

(g) identifying different versions of pattern rules (leads to equivalence of expressions – number sentences with no equals), and justifying why it works for all terms;

(h) using pattern rules to introduce variable and algebraic expression; and

(i) representing patterns with graphs and relating growing and fixed parts to slope and $y$ intercept of graph respectively.

Students gain a better understanding of patterning if given experience finding number pattern rules without using a table. An example of this is below.

**Example:** Consider the following:

<table>
<thead>
<tr>
<th>$X$</th>
<th>XX</th>
<th>XXX</th>
<th>XXXX</th>
<th>XXXXXX</th>
</tr>
</thead>
<tbody>
<tr>
<td>XX</td>
<td>XXX</td>
<td>XXXX</td>
<td>XXXXX</td>
<td>XXXXXX</td>
</tr>
</tbody>
</table>

The fixed part is the three $X$’s from term 1, the growing part is two extra $X$s each new term. In a table it is easy to see that term 1 is 3, term 2 is 5, term 3 is 7 and so on, leading to a pattern of $2n-1$ for the $n$th term. However, if we stay with visuals, then more is possible. The visuals can be interpreted as

- two rows, the top is $n$ and the bottom is $n+1$, making the pattern $n+n+1$;
- a double row of length $n$ and an extra $X$, making the pattern $2n+1$; and
- a double row of length $n+1$ with a missing $X$, making the pattern $2(n+1)-1$.

The focus on visuals gives the students an understanding that there are different equivalent algebraic expressions for a pattern rule. The different interpretations of the visuals also provide arguments to support that the pattern rule holds for all items; they provide justification.

The common materials to be used consist of numbers or objects of different colour, size, shape, etc., with the number of objects (or the numbers themselves) determining the basis of the pattern, plus small numbered cards (1 to 5), tables, and calculators. Again it is useful to have magnetic copies of objects, cards and numbers.

The major ideas to be developed are: (a) generalisation and justification of generalisation, (b) visual manipulation and numerical tabulation, (c) representation of generalisations with variables (introduction to algebra), and (d) introduction of line graphs and relation of slope and $x$-intercept to growing and fixed parts of growing patterns. As well, the difference between use and non-use of number tables can be identified; number tables make identification of position pattern rule easier but determining the rule from visuals alone enhances students’ ability to justify their rule and to find more than one version of the rule (which helps develop equivalence of expressions).
Patterns in other mathematics strands

Patterns in other strands can be used to obtain understanding of ideas and recall of facts in these strands. For example, the order of place value positions, the relationship between adjacent place-value positions, counting patterns and the odometer principle, multiplication basic facts, higher decade facts (3+4=7 \rightarrow 30+40=70) and repeated addition are all examples of understandings and facts that can be obtained by seeing patterns.

Much of this is covered in the *Teaching Number* and *Teaching Operations* resource books.

2.2 Very early patterning activities

Patterns are an excellent way to teach students the act of generalisation and to introduce variables, algebraic expressions, and graphs. There are two types of patterns, repeating and growing; and three types of pattern rule to identify – the repeating part for repeating patterns, and the sequential and position rules for growing patterns. Materials used are objects with different attributes (e.g., size, shape, colour) and cards with numbers 1, 2, 3, 4 and 5 on them; and a teacher set of these materials magnetised for placing on magnetic whiteboard.

Copy, continue and create repeating patterns

The steps here are:

(a) Use movements and sound (e.g., body movements, clapping, music and dance steps) to follow patterns of activity.

(b) Extend these movement and sound activities to where students have to copy and continue a pattern that repeats.

(c) Encourage students to create their own patterns of movements and sound and lead others in copying their patterns.

Copy, continue, complete and construct a recorded repeating pattern

The first activity is to copy a pattern (e.g., O X X O X X X ... where O - red block and X - blue block. Teacher places out the pattern with materials and students build a copy.

The second activity is to continue a pattern (e.g., O X X X O X X X O ...). Teacher starts the pattern and student continues it (e.g., X X X O X X O X ...). It seems to be a better teaching sequence and easier for students if they copy patterns before continuing them.

The third activity is for the student to identify the repeating part. Students often identify only part of the repeating part (e.g., X X not O X X), so language must direct students to find all of the repeating part.

The fourth activity is to complete a pattern – to fill in the empty spaces (e.g., what shapes will fill the gap in X X O O O X X O O O X X O ____ X O O O ...)

The fifth activity is to have the students construct their own patterns and identify the repeating part. Here students sometimes construct a symmetric design which does not go linearly on forever (e.g., X X X O O X X O O X X O ...) not a continuing repeating pattern (e.g., X X O O X X X O O ...).

Finally the sixth activity has the student reversing the direction of the previous activities and constructing a repeating pattern when given a repeat such as O O X X X (e.g., O O X X X O O X X X O O X X X ...).
Notes: (a) The example in the final activity can be made much more difficult if the repeat given is O X X O. This is because the repeating part starts and finishes with the same object – always begin with something simpler that repeats to do this. (b) All examples above have only 2 objects (X and O). Three or more can be used for more difficult repeating patterns. (c) Young children find it easier to work with repeating patterns if the objects are very different. This often means two attributes different – both colour and shape (e.g., red O, blue X).

Determining what object is in a position in a repeating pattern

In this activity, the teacher provides the start of a repeating pattern (e.g., X O O X O O ...) and numbers each object as below.

\[
\begin{array}{ccccccc}
X & O & O & X & O & _ & _ \\
1 & 2 & 3 & 4 & 5 & 6 & 10
\end{array}
\]

The teacher then asks the students to identify the object (X or O) that is in a particular position, e.g., the 10th position as in the example above. The sequences involved in this are:

(a) initially allow the students to put out extra objects until the position is reached before asking them to work it out in their head;

(b) initially give the students positions to find that are 5 or less ahead of the last item placed before going to positions further out (e.g., finding the 10th position is easier than the 13th position); and

(c) initially allow students to copy the pattern when they are asked to determine the object in positions before asking them to find a position in a pattern that the teacher has put out.

It seems that determining what object is in a position requires students to coordinate two things in their minds, the pattern and the position number, or, more difficult, synchronise these two things as moving, in their mind, along the pattern of objects and along the number for the position of the objects. It also requires the students to identify the whole repeat and recognise its components (e.g., two Xs and one O). This is easier to do if the students can put out the objects as they go, the number for the position past the last object placed is within the students’ subtitisation range (normally less than or equal to 5), or the students have familiarised themselves with the pattern by placing it out themselves.

As students get older and gain skip counting proficiency or improved understanding of multiplication, they can use the pattern to find the object and so larger numbers can be set as below, e.g., find the 27th term

\[
\begin{array}{ccccccc}
X & O & O & X & O & _ & _ \\
1 & 2 & 3 & 4 & 5 & 6 & 27
\end{array}
\]

The students can determine that, for example, the 27th term is a O because they see that X O O ... is a pattern of three and 27 as a multiple of three must be the last object in the pattern of three (i.e., the students can jump in 3s). As a beginning to this, students start to manipulate the objects emphasising features as below – they start to think of the pattern in terms of its last object.

\[
\begin{array}{ccccccc}
X & O & O & X & O & _ & _ \\
3 & 6 & 9 & 27
\end{array}
\]
Sometimes, the students use a wrapping technique as below – here the students appear to be using multiples of 6 not 3 and seeing 27 as 3 more than 24 and counting half-way along the 6 objects.

\[
\begin{array}{cccc}
X & O & O & X & O & O \\
X & O & O & X & O & O \\
\hline
\end{array}
\]

Because skip counting by 5 and the 5 times tables are more familiar to students, 5 object patterns are easier than 3 or 4 object patterns, e.g., students find that a pattern like 0 0 X X X ... is easier than a pattern like O X X X ... .

Finally, students seem to find patterns like O O X ... easier to find objects in positions than patterns like X O O ... because the students can tag the 3rd or repeat ending object.

**Note:** Determining what object is in what position is difficult and can also be part of patterning when growing patterns are being developed. It requires students to: (a) co-ordinate (synchronise) counting and pattern; (b) identify the whole repeat and the number of objects in it and relate this to the position of the object to be found; and (c) tag the last term and use this for skip counting towards term.

### 2.3 Early to middle patterning activities

#### Moving from repeating to growing patterns

The first step in this is to break the pattern into repeats. This assists the students to identify the repeating part and to determine object for positions (see activity (3) above).

The first step is for the teacher to set up a repeating pattern; say X O O X O O X O O ... .

The second step is to ask the students to identify the repeating part and then to break the pattern into repeats and to separate the repeats as below. Note that when separating the repeats it can be useful to discuss and trial other ways of representing the repeat, as below.

The third step is to get the students to construct a set of number cards and to place these under the repeats as below. **[Note:** This means that repeating patterns have the same term structure as growing patterns but they do not grow.]

The fourth step is to ask students to pick one of the objects and grow it, as below.
The teacher can provide a variety of activities – grow one object, grow both objects, grow both at different rates, change the way the objects are presented, and so on (see example on right). This presents repeating patterns as a precursor to growing patterns and links the two together.

**Copying, continuing, completing and constructing growing patterns**

Once growing patterns are introduced, students can be asked to: (a) copy, continue, and complete growing patterns set up by the teacher; and (b) create their own growing pattern and explain how it is growing. It is important to use a variety of objects and to build the patterns using a variety of attributes (e.g., colour, size, shape).

This activity is the major one in the middle years. It is similar to the steps that were in repeating pattern activities. The first step is for the teacher to show a growing pattern up to the 3rd term, as on the right.

Second step is for the students to copy this – to make their own copy.

The third step is for the students to continue this pattern for the next few terms, for example this could involve making the 4th and 5th terms as in the diagram above. It is useful for students to have number cards and place these under the terms.

The fourth step is to ask the students to make some further terms (or tell what is involved in these terms) such as the 7th, 10th or 20th term. This requires some understanding of the pattern rule but can be completed by considering what happens term by term.

The fifth step is to ask the students to make the first 5 terms of their own growing patterns and to explain what is involved in the patterns and how the terms are growing.

The sixth step is to complete a pattern – this is where there are gaps in the example given, as on the right. It should be noted that it is more difficult to complete than continue a pattern. It is also harder if the terms given are not regular, e.g., when you are given the 1st, 2nd and 5th terms. [Note the example on the right has X increasing by one and O increasing by two in each new term.]

**Determining growing and fixed parts and finding and using pattern rules**

The important part of growing patterns is to identify the general rule for the pattern that enables any term to be determined – this is called the pattern rule. Determining pattern rules is assisted by identifying in the pattern what grows and what stays fixed – the growing part and the fixed part rule.

The first step is to look at patterns and discuss what grows and what is fixed. It is important to start this process with simple growing patterns such as that below.
The second step is to use the above information to be able to work out what each term will look like. This is usually done for some numbers and then considered for any term. For the example on right, the 10th term is one star and 10 circles, the 20th term is one star and 20 circles, and so on.

The third step is to identify the pattern rule. There are two types of pattern rule – the sequential pattern rule which gives the difference between sequential terms, and the position rule which relates number of objects to the term position. The sequential rule for the pattern above is “add one X”, and the position rule is 1+ position number. We should accept messy language; later we can give the rule in terms of n.

The fourth step is application to real world problems. For example, a table in a restaurant sits 4, two tables pushed together sits 6, and so on. How many will 15 tables pushed together into a row sit?

The fifth step is to reverse the process and find the position of the term with a given number of objects. For example, in the pattern above, what position has 62 objects? [the 61st position]. Another example is how many tables have to be pushed together in a row to sit 18 people?

The sixth step is to teach students to solve patterns both visually and by use of tables. To assist with generation of rules from visual cues, it is important to teach students to visualise objects in different ways. Consider the pattern on the right.

The pattern consists of two towers and these can be viewed in at least three ways. A sees one double tower and an extra, B sees two single towers and C a double tower with one missing.

The three cases have the same sequential rule which is “add 2”. However, each case has a different but equivalent position pattern rule. For the third term, A is a 2 x 3 tower plus an extra 1, B is a 3 tower plus a 4 tower, and C is a 2 x 4 tower minus 1. Thus, for any term or position, A gives the position pattern rule of “2 times the position plus 1”, B gives “position plus position plus 1”, and C gives “2 times position plus 1 minus 1”. In algebra these are 2n+1, n+n+1, and 2(n+1)-1 for any n – which are the same thing.
Tables versus visuals for growing patterns

There has been debate over when tables of numbers should be used as a way of finding position pattern rules, as against determining the pattern from the visuals. It is important that students be taught all strategies; however, for growing patterns in numbers, tables are the main strategy (as below).

<table>
<thead>
<tr>
<th>Number</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>9</th>
<th>and so on</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

This booklet recommends that both the visual and table strategies be taught. Therefore, in the second half of the middle years, introduce tables to scaffold thinking as below.

**Step 1:** Teacher provides a growing pattern as on right.

**Step 2:** Students construct and complete a table as below right.

**Step 3:** Students find the sequential and position pattern rules. For the sequential rule, look down the table; for the position rule, look across the table. Looking down the right hand column, the sequential rule can be seen as “add 2”. Looking across the table, the position rule is found by checking if there is some multiplication and/or addition/subtraction that works for all numbers. For example, 1×2 = 2 which is 1 more than 1, 2×2 = 4 which is 1 more than 3, so try this on next (yes, 3×2−1 = 5, 4×2−1 = 7 and so on). Thus, the position rule is that, for a number like 256, the number of objects is 2×156−1, and in everyday language, as a generalisation, it is “2 × term number − 1”.

(Note: It is a good idea to ask the students what it would be for any position number \( n \) but not expect all to get the answer as an algebraic expression.)

**Step 4:** Reverse the direction by giving students a rule such as “3 times the position plus 2” and ask them to construct a pattern for this rule. Do the same for a sequential pattern rule.

Note: The table is simpler to use to find the position rule. However, the visual method gives the reason for the rule and gives more than one rule.

Tables in repeating patterns

It is also possible to use repeating patterns to develop generalisation. To do this, a repeating pattern is displayed, the students are asked to break it into repeats, and then complete a table for these repeats from which generalisations can be found. An example of this is:

| Pattern: | XXOXXOXXO.....; |
| Repeats: | XXO XXO XXO ; |
| Table: as on right |

<table>
<thead>
<tr>
<th>Number of repeats</th>
<th>Number of Xs</th>
<th>Number of Os</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

**Generalisations:** Look for “patterns” or “rules” down and across the table: the down generalisations include: repeats go up by 1, Xs go up by 2, Os go up by 1, and total goes up by 3; and the across generalisations include number of Xs is twice the number of repeats, number of Os is equal to the number of repeats, and total is three times the number of repeats.
2.4 Later patterning activities

Building algebraic generalisations from growing patterns

In the later years, it is important to ensure that students can find and express generalisations from patterns in quasi-generalisation form (for large numbers – e.g., $3 \times 674 + 2$), in language (informally – e.g., “3 times the position plus one”), and in algebraic form (e.g., $3n + 2$). This is developed in the following sequence.

**Step 1:** Present a pattern as on right, e.g., how many sticks to make, 1, 2, 3, ..., squares in a row

**Step 2:** Identify the fixed and growing parts of the pattern (the first stick is fixed and then it grows by three sticks, as on right).

**Step 3:** Determine the sequential rule (e.g. “add 3”) and the position rule (number of sticks is 3 times position plus 1). Note: To assist finding the positional rule, do the following:

- record the number of sticks for each term – term 1 is 1+3 = 4 sticks, term 2 is 1+3+3 = 7 sticks, term 3 is 1+3+3+3 = 10 sticks, ...
- develop quasi generalisation – e.g., for term 7, number of sticks is $1+3\times7 = 22$, for term 100, number of sticks is $1+3\times100 = 301$, for term 345, the number of sticks is $1+3\times345 = 1036$;
- state the generalisation in language – “1 stick at start plus 3 for every term”.

**Step 4:** Put the generalisation in algebraic terms (any number represented by the letter $n$) as $1+3n$.

**Step 5:** Use the generalisation to reverse the position-to-number activity to number-to-position, e.g., how many squares use 28 sticks? [Position 9 has 1+3×9 sticks, so it is 9 squares.]

**Step 6:** Reverse everything and construct a pattern from a pattern rule, e.g. $2n+3$ (as on right).

The activity can be extended to graphing. Once the position pattern rule has been determined as an algebraic expression, a graph can be constructed. To do this, continue as follows.

**Step 7:** Construct a table and place early values in it, e.g.,

<table>
<thead>
<tr>
<th>Number of sticks:</th>
<th>4</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
<th>and so on</th>
</tr>
</thead>
<tbody>
<tr>
<td>Position:</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td></td>
</tr>
</tbody>
</table>

**Step 8:** Use the table to draw the graph (as on right).

**Step 9:** Reverse everything:

- provide a graph as on the right; and
- make up a pattern to match this graph (use the fact that term 1=4, term 2=6, term 3=8 to construct a pattern – see below).

**Note:** There is a relationship between graphs and patterns that holds generally for growing parts and, in a way, for fixed parts. The slope of the graph is the growing, and the $y$ intercept is the fixed part of a growing pattern. It works on the two examples in (10) and (11). The (10) graph (see step 2 of (10)) grows by 3 and has a fixed part of 1 – this is seen in the graph as a slope of 3 and a $y$ intercept of 1. The (11) graph (see step 8 of (11)) grows by 2 and has 2 fixed – its graph has slope of 2 and a $y$ intercept of 2.
Building algebraic generalisations from repeating patterns

The previous sequence can be extended to use repeating patterns as a means of teaching generalisation to algebraic expressions (as is possible in growing patterns) as follows.

**Step 1:** Teacher presents a repeating pattern – we’ll use the following pattern: X X O X X O X X O ... XXO XXO XXO ... in repeats.

**Step 2:** Complete the table below and find the down and across patterns.

<table>
<thead>
<tr>
<th>Number of repeats</th>
<th>Number of Xs</th>
<th>Number of Os</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>3</td>
<td>9</td>
</tr>
</tbody>
</table>

**Step 3:** Generalise how to find the numbers for \( n \) repeats. Do this in language first – Os are the same as number of repeats, Xs are double number of repeats, and so on.

**Step 4:** State generalisations as algebraic expressions – if \( n \) repeats, number of Xs is \( 2n \), number of Os is \( n \), and total number is \( 3n \).

**Step 5:** Reverse the repeats to number activity (to number to repeats) – how many repeats are needed for 26 Xs? [26=2n so it is 13 repeats.]

**Step 6:** Reverse the activity algebraically (e.g., if \( n \) Xs, then \( n/2 \) repeats).

**Step 7:** Graph the relationship between number of Xs and number of repeats.

**Step 8:** Reversing overall – construct a repeating pattern of Xs and Os for a given graph, e.g., the graph below. This graph has 3 Xs for 1 repeat, 6 Xs for 2 repeats, so it is OXXX, OXXX ... or OOXXXOXXXX ....
Extending repeating patterns to fractions and ratio/proportion

The use of repeating patterns is only the start – repeating patterns are excellent for introducing fractions and ratios and equivalent fractions and ratios, as follows.

**Pattern:** Start with a repeating pattern, e.g., X X O O O X O O O X X O O O

**Repeats:** Change it to repeats, e.g., XXOOO XXOOO XXOOO

**Table:** Complete a table that contains fractions and ratios.

<table>
<thead>
<tr>
<th>Repeats</th>
<th>Xs</th>
<th>Os</th>
<th>Total</th>
<th>Fraction Xs</th>
<th>Ratio X:O</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>2/5</td>
<td>2:3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>6</td>
<td>10</td>
<td>4/10</td>
<td>4:6</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>15</td>
<td>6/15</td>
<td>6:9</td>
</tr>
</tbody>
</table>

**Generalisation:** Look at what \( n \) repeats will give (repeats to Os and Xs).

**Reversing:** Go from numbers of Os and Xs to repeats (e.g., 72 Os means 72/3 repeats = 24 repeats)

**Equivalence:** Discuss whether fractions and ratios are equivalent (or in ratio terms, in proportion). For example, is 2:3 = 4:6?

**Note:** A student explained it to class as shown on right. She stated that 2:3 is XXOOO and 4:6 is XXXXOOOOOO. However, she rearranged the objects as on right and argued that this showed that XXXXOOOOOO is the same as XXOOO.

**Reversing overall:** Make up a repeating pattern where the ratio is 2:5 (e.g. O O X X X X O O X X X X O O X X X X and so on)

**Note 1:** The teaching approach is to have class discussion. Let students propose generalisations. Don’t say if generalisations are right or wrong. Focus on down and across generalisations. Encourage students to justify their point of view.

**Note 2:** Repeating patterns can be done with number, e.g., 1 2 2 1 2 2 1 2 2, ..., but numbers here act like objects.
Algebra can reveal the structure of mathematics so that students can work from the whole to the part (see section 1.4). There are two important principle types to be revealed in arithmetic that also apply to algebra. They are made up of principles that apply to numbers, variables, expressions and functions. They are the equivalence and order structure and the field or operation structure. There is a third principle type that has to do with number-size – this is described in the YDM Teaching Operations resource. The sequence for teaching the principles is as follows.

![Figure 5. Sequence for teaching Arithmetic Principles]

### 3.1 Equivalence, order and Field structure

**Equivalence and order**

Arithmetic and algebraic expressions, whether they have one or many symbols, can be equal to each other (equivalence) or greater or less than each other (order).

When arithmetic examples of equivalence or order (e.g., 2+3=5 or 4+3>6) are studied, generalisations can be found. As will be shown in equivalence and equations (see section 2.4), this is best done in unnumbered situations initially. These lead to structures made up of generalisations which we call principles.

The *equivalence* principles are as follows:

1. **Reflexivity principle.** Anything equals itself (e.g., 2 = 2).
2. **Symmetry principle.** Equals can be turned around (e.g., if 2+3 = 5 then 5 = 2+3).
3. **Transitivity principle.** Equals continues across equal relationships; the first in a sequence, equals the last (e.g., if 2+3 = 5 and 5 = 6−1 then 2+3 = 6−1).

The *order* principles are as follows:

1. **Well-ordered principle.** Two expressions have to be equal to, greater than or less than each other (e.g., 3 , 5 → 3 < 5 23+64, 72−15 → 23+64 > 72−15).
2. **Antisymmetry principle.** If two expressions that are less than are turned around then they become greater than (e.g., 6 < 9 is the same as 9 > 6).
3. **Transitivity principle.** Order continues across relationships, the first in sequence is greater than / less than the last (e.g., $6-1 < 8+2$, $8+2 < 16-4$, $16-4 < 48\div2 \rightarrow 6-1 < 48\div2$).

The methods for teaching them relate (as in other sections) to the use of models to show equals and order. The two most used are mass (balance) and length (strips of paper/number lines). Section 3.2 contains instructional activities for teaching the equivalence and order principles.

**Field or operation**

The most important structure in school mathematics is the field. It is a structure composed of expressions and two operations, addition and multiplication. (Note: subtraction and division are not strictly operators as they do not obey all operation principles.)

The field or operation principles are as follows.

1. **Identity principle.** There are two identities that leave everything unchanged, namely, 0 for addition and 1 for multiplication (e.g., $27+0 = 27$ and $1\times3 = 3$).

2. **Inverse principle.** If a change is to be made it can be undone by an inverse change, namely, $-3$ for $+3$ and $\div7$ for $\times7$ (Note this means that $-$ is the inverse of $+$ and $\div$ is the inverse of $\times$).

3. **Commutative principle.** Operations with $+$ and $\times$ can be “turned around” without error. (e.g., $6+7 = 7+6$ and $23\times4 = 4\times23$). However, this is not true for $-$ and $\div$ (e.g., $6-2 < 2-6$ and $12\div3 \neq 3\div12$).

4. **Associative principle.** Operations with $+$ and $\times$ which have more than 2 expressions can be completed with any association in any order. (e.g., $6+3+5 = 9+5$ or $6+8$ or $11+3$).

5. **Distributive principle.** Addition means adding “like things” but multiplication means multiplying everything, that is, $\times$ distributes across $+$ (e.g., $3\times(4+5) = (3\times4) + (3\times5)$; $23\times2 = (20\times2) + (3\times2)$). This means that $23+3 = 26$ but $23\times3 \neq 29$, and $23\times3 = 69$ but $23+3 \neq 56$.

There are additional principles as follows that emerge from these (but are well worth remembering in their own right).

1. **Compensation principle.** If there is a change in one number in $+$ or $\times$, this is compensated by the inverse change in the other number (e.g., $8+5 = 10+3$ ($8+2 = 10$, $5-2 = 3$); $12\times3 = 4\times9$ ($12\div3 = 4$, $3\times3 = 9$)). Because of their inverse nature, $-$ and $\div$ also compensate but not by using inverses; they compensate by doing the same to both numbers (e.g., $8-5 = 12-9 = 3$ ($8+4 = 12$, $5+4 = 9$); $12\div3 = 24\div6 = 4$ ($12\times2 = 24$, $3\times2 = 6$)).

2. **Inverse relation principle.** For $+$ and $\times$, any increase in a number, increases the total (e.g., $8\times4 = 32$, $8\times6 = 48$). However, for $-$ and $\div$, increasing the second number decreases the total (e.g., $8-3 = 5$, $8-6 = 2$ (3 increases so 5 decreases); $12\div3 = 4$, $12\div6 = 2$ (3 increases so 4 decreases)).

3. **Equivalence principle.** Since $+0$ and $\times1$ do not change anything, then two things are equivalent if one is $+0$ or $\times1$ of the other (e.g., $2\times1 = 2\times2/2 = 4/6$, so $2/3$ is equivalent to $4/6$ because $2/2$ is the same as 1; $404-186 = 404-186+0 = 404+14-186-14 = 418-200$, so $404-186$ can be solved by $418-200$ because $+14-14$ is the same as $+0$).

Similar to before, these principles can be taught by the models that represent $+$ and $\times$; the set and number line for $+$ and the set, number line and array for $\times$. Section 3.2 contains instructional activities for teaching the operation or field principles.
3.2 Very early Arithmetic Principles activities

Arithmetic principles reflect the structure of arithmetic and algebra. If they are learnt, then students have knowledge they can apply across all years of primary and secondary school mathematics – to whole numbers, fractions, variables, functions and calculus.

There are two major structures that the principles can be clustered into: the equivalence and order structure, and the field or operation structure. It is difficult to state where in the school years the principles should be developed because they need to be reinforced every time a new context (new types of numbers, variables, functions, etc.) appears.

The materials that are commonly used to teach these principles are those associated with the areas of equations and functions, namely, the balance and the length models, and those associated with the two operations of addition and multiplication, namely, the set, number line and array models. However, calculators and finding patterns are also excellent instructional activities for this area, and there are some active whole-body techniques that are very useful. It has also been found that activities in unnumbered situations (situations where there is no number) assist in seeing these structures.

The early years are when the balance and length models are introduced. If unnumbered contexts like groceries and coloured strips of paper are used for these models, the equivalence and order principles can be introduced.

Teaching the equivalence principles – unnumbered contexts

The equivalence principles are reflexivity, symmetry, and transitivity. They can be taught with the balance (mass) and length (strips of paper) models.

Balance or mass model.

Here equals is represented with groceries on a balanced beam balance. (Note: The use of balances begins as a real balance in early activities and then abstracts to a mathematical-balance drawing in middle activities and finally to an image of a balance in later activities. The drawing and the image balance can allow any operation.)

The principles are easily shown physically with a balance. So have the students experience the principles with beam balances and groceries. Direct the students to state out loud the equation as you move your hand from their left hand side of the balance to their right hand side – “equals” is said as the hand goes past the balance point (centre) of the balance (Note that it can be useful to stick “=“ on the centre of the balance). Direct the students to record the balanced groceries as informal equations, e.g., “salt equals soap plus pasta”. Discuss any generalities they find; encourage them to see the generalities below.

1. Reflexivity: This is fairly obvious (that 2 things that are the same will be equal) but it needs to be made explicit with equations. Students can investigate if same things always balance?

2. Symmetry: This can be seen by turning the balance 180°.
3. **Transitivity:** This requires the students comparing three things in three different ways to show \( A = B \) and \( B = C \) means \( A = C \).

![Diagram showing transitivity]

**Length model.**

Here equals and order are represented by same length and different length respectively. Again it begins by strips of paper, moves onto lines (the double number line) and finally to images in the mind. Again the equivalence principles are easily represented as in the 3 examples. Once again, effective strategies involve the students saying out loud the equalities, writing informal equations, and discussing generalities.

1. **Reflexivity:**

   \[
   \begin{array}{c}
   A \\
   \hline
   A + B = A + B
   \end{array}
   \]

2. **Symmetry:**

   \[
   \begin{array}{c}
   A \\
   \hline
   A + B = C
   \end{array}
   \]

3. **Transitivity:**

   \[
   \begin{array}{c}
   P = Q \\
   \hline
   P = R + S
   \end{array}
   \]

   Note: It is always best to start with (a) **human body** (e.g. hang plastic bags on arms and become a beam balance and walk different distances); and, as stated before, (b) in **unnumbered** situations (e.g. compare groceries, use unmeasured strips of paper).
Teaching the order principles – unnumbered contexts

The order principles are well ordered, antisymmetry and transitivity. They can be taught with the balance (mass) and length (strips of paper) models. (Note – for the balance, the heavier item pushes down more, so higher is lighter). Once again, effective strategies involve the students saying out loud the equalities, writing informal equations, and discussing generalities. Drawings now show how both balances with weights and paper strips can give students experiences with the order principles on which to base discussion.

1. **Well ordered**: It is obvious that if you have two weights or two strips that they either have to be equal or one is larger than the other. However, this needs to be experienced and made explicit.

- \( A = B \) or \( A > B \) or \( A < B \)

\[
\begin{array}{c}
P = Q \\
or \\
Q < P
\end{array}
\]  

2. **Antisymmetry**: As for equivalence and symmetry, this can be seen by turning the balance 180°.

\[
\begin{array}{c}
A > B \\
\Rightarrow B < A
\end{array}
\]

3. **Transitivity**: Once again the students compare three things in three different ways to show \( A=B \) and \( B=C \) means \( A=C \).

\[
\begin{array}{c}
P > Q \\
and \\
Q > R
\end{array} \quad \Rightarrow \quad P > R
\]
3.3 Early to middle Arithmetic Principles activities

The middle years are when the equivalence and order principles are reinforced in numbered situations and when the field/operation principles are introduced. The equivalence and order principles rely on balance and length models, while the field/operation principles are taught by the models that represent addition and multiplication, the set and number line for addition and the set, number line and array for multiplication.

Introducing the equivalence and order principles for numbered situations

In the middle years, the equivalence and order principles (from (1) and (2) above) are applied to numbers using balance beams with numbers of weights (e.g., coat-hanger balances with red, blue and green baked bean cans) and lines of blocks. (Note: further work to reinforce these concepts should be undertaken with drawings of mathematical balances and vertical double number lines in the latter half of the middle years.)

We will not attempt to show how to teach each principle in detail but provide diagrams of how activities with balances and number lines can be experienced and recorded as a basis for discussion and recognition of generalisations. We will not include reflexivity as it is obvious and transitivity will cover both equivalence and order.

1. **Symmetry**: This is important as it shows that an equation can be reversed, e.g., $3+4=7$ and $7=3+4$ are the same.

   ![Symmetry Diagram]

2. **Antisymmetry**: This extends symmetry in equivalence to order showing that when inequations are reversed the order changes from greater than and less than and vice versa.

   ![Antisymmetry Diagram]

3. **Well ordered**: This is making explicit the principle that two expressions are either equal or in order (less than and greater than).

   ![Well Ordered Diagram]
4. Transitivity: This is showing that if one thing is equal to/less than/greater than a second thing and this second thing is equal to/less than/greater than a third, then the first is equal to/less than/greater than the third.

\[
P = Q \quad \text{and} \quad Q = R + S \quad \rightarrow \quad P = R + S
\]

Introducing the field/operation principles

The field/operation principles comprise identity, inverse, commutativity, associativity, distributivity, compensation, inverse relation, and equivalence. There are two ways to teach the principles. The first is to use the set, array and number-line models to show the relationships. The second is to simply use a calculator to check many possibilities.

As for equivalence and order principles, effective strategies for teaching the field/operation principles involve the students (a) experiencing activity with models, (b) saying out the relationships experienced on the way to the principles, (c) writing informally what these relationships are, and (d) discussing generalities. As well, any arithmetic experiences should be used to highlight the principles as they appear. The models used are the set, number line and array.

There is not the space to show in detail how to teach each principle, so diagrams will show how models may be used to teach them. In some cases, interesting and effective methods will be highlighted.

1. Identity: The aim is to show that adding 0 and multiplying by 1 do not change anything. For addition, the best idea is to add 2, then add 1 and finally add 0. For multiplication, we have 1 group, row or jump of the number or a number of groups, rows and jumps of 1 (both of which equal the number).

2. Inverse: The aim is to show that addition and subtraction and multiplication and division are inverses. Act out situations like “Share 12 amongst 3, what do we get?” [4]. “Make 3 groups of this, what do we get? [12]”. Join and separate, rejoin and reseparate; make groups and share out groups, remake and reshape. Highlight that one action is the opposite and undoes the other (e.g., 7+2=9 and 9-2=7).
3. **Commutativity** ("turnarounds"): Show that order does not matter for addition and multiplication. This is achieved by showing two numbers added gives the same regardless of the order, and (highly recommended) turn an array by 90 degrees to show, e.g., that 3 rows of 4 is the same as 4 rows of 3.

![Diagram of 7 joined by 2 and 2 joined by 7]

4. **Associativity**: The aim is to show that order does not matter for three numbers for the same operation of addition or multiplication. This is straightforward for addition (joining 3 sets so which sets join first is irrelevant – see below), but not so easy for multiplication because you need (3 groups of 4) groups of 5 objects to equal 3 groups of (4 groups of 5). This is more easily seen with calculation (and a calculator) than from the models.

![Diagram of 3 + 4 + 5 = (3 + 4) + 5 = 3 + (4 + 5)]

5. **Distributivity**: This is best done by dividing arrays, or the more abstract rectangles, into two parts.

![Diagram of 7 by 4, 5 by 4, and 2 by 4]

6. **Compensation**: This is where we show that any two numbers have the same addition or multiplication if a change in one number is undone in the other number by use of inverse. It is difficult to show with models. In fact, it is sometimes more easily seen by discussion, e.g., “look at 2+3=5, what happens if 2 goes to 4, what if we want the sum to remain as 5?” However, models can be used as below.

![Diagram of 4 + 7 = 4 + 2 + 5 = 6 + 5 and 4 x 6 = 2 x 2 x 6 = 2 x 12]

**Note**: This is one area where using kinaesthetic or whole body activity is useful. Students have difficulty looking at materials and pictures and seeing that, if increasing the 8 in 8+5=13, it is necessary to decrease the 5 to keep 13 as the answer. However, one effective way to overcome this
difficulty is to consider addition as a relay race in which one member does more than their share, and to act this out. Get children to form into pairs, mark out a relay walk (as on right) and a baton change and direct the pairs to walk the relay.

Discuss what would happen if the 1st person walked further (as below right) – what happens to the 2nd person?

Students can see that the 2nd person has to walk less by the amount the 1st person walked more.

This method of teaching appears to make it easier for students to understand the compensation principle.

**Extra note.** It should be noted that although the field/operation principles apply to addition and multiplication, there is also compensation for subtraction and division. However, it can be confusing for students because compensation for addition/multiplication is the inverse of the first change but compensation for subtraction/division is the same as the first change. For example, 8+5=10+3 because, for addition, 8 increased by 2 means that 5 decreased by 2; but 8-5=10-7 because, for subtraction, 8 increased by 2 means 5 increased by 2. Similarly, 12×4=6×8 because, for multiplication, 12 divided by 2 means 4 multiplied by 2; but 12÷4=6÷2 because, for division, 12 divided by 2 means that 4 is divided by 2.

To avoid confusion either teach compensation for addition and multiplication, and subtraction and division separately, or place compensation under the inverse principle. Since addition and subtraction, and multiplication and division, are inverses, then it is reasonable that they would do the opposite with regard to the Compensation principle. Since addition and multiplication compensation require the opposite change, it is reasonable that subtraction and division compensation require the opposite of opposite which is the same change.

7. **Inverse relation:** Subtracting more and dividing by more, decrease the computations, as can be seen in the examples below.

   ![Diagram](image)

   The more that is taken away, the less that is left.

8. **Equivalence:** This principle says that as long as add 0 and multiply by 1, the answer stays the same (the expressions are equal). This is required to be experienced and for students to become flexible with what 0 and 1 could be. For example, for 28+15, 0=+2−2, which means that 28+15 = 28+15+2−2 = 30+23, while for 2/5, 1 = 2/2, which means that 2/5 = 2/5×1 = 2/5×2/2 = 4/10.
Reinforcing the equivalence and order principles with more abstract representations

At the end of the middle years, the equivalence and order principles can be further reinforced with mathematical balances (drawings with expressions on each side) and vertical double number lines (with operations drawn as arrow movements on left and right of the vertical line. We provide a few illustrations of the kind of activities students can experience.

Symmetry

\[
\begin{align*}
2x3+5 & \quad 20-9 \\
\downarrow & \quad \downarrow \\
2 \times 3 + 5 & = 20 - 9 \\
\end{align*}
\]

\[
\begin{align*}
20-9 & \quad 2x3+5 \\
\downarrow & \quad \downarrow \\
20 - 9 & = 2 \times 3 + 5 \\
\end{align*}
\]

Antisymmetry

\[
\begin{align*}
2x3+5 & \quad 20-3 \\
\downarrow & \quad \downarrow \\
2 \times 3 + 5 & < 20 - 3 \\
\end{align*}
\]

\[
\begin{align*}
20-3 & \quad 2x3+5 \\
\downarrow & \quad \downarrow \\
20 - 3 & > 2 \times 3 + 5 \\
\end{align*}
\]

Well-ordered

\[
\begin{align*}
2x3+5 & \quad 20-9 \\
\downarrow & \quad \downarrow \\
2 \times 3 + 5 & = 20 - 9 \\
\end{align*}
\]

\[
\begin{align*}
20-9 & \quad 2x3+5 \\
\downarrow & \quad \downarrow \\
20 - 9 & = 2 \times 3 + 5 \\
\end{align*}
\]

\[
\begin{align*}
2x3+5 & \quad 20-6 \\
\downarrow & \quad \downarrow \\
2 \times 3 + 5 & < 20 - 6 \\
\end{align*}
\]

\[
\begin{align*}
20-6 & \quad 2x3+5 \\
\downarrow & \quad \downarrow \\
20 - 6 & > 2 \times 3 + 5 \\
\end{align*}
\]
3.4 Later arithmetic principles activities

In the later years, the principles need to be reinforced for number and then applied to variables. This should be seen as an extension of number work.

Symbolically reinforcing the equivalence, order and field/operation principles

An effective way to reinforce the principles for numbers is to give students calculators and encourage them to explore principles (e.g., Is the 1st number plus the 2nd always equal to 2nd plus 1st? Is a number multiplied by 1 always equal to itself?). A more structured way to do this is given for the distributive law.

- provide students with examples to check with the calculator, e.g.,

Are these the same?

\[11 \times 13 + 11 \times 24 = \quad ?\] ; \[11 \times (13 + 24) = \quad ?\]
\[23 \times 27 + 23 \times 56 = \quad ?\] ; \[23 \times (27 + 56) = \quad ?\]
and so on

- give students examples to solve that require using the principle being checked, e.g.,

Calculate these without adding the two numbers in the bracket!

\[54 \times (76 + 28) = \quad ?\] 
\[186 \times (259 + 543) = \quad ?\] 
and so on

Extending the equivalence, order and field/operation principles to variables/algebra

An effective way to show that the principles apply to algebra is to build the principles from arithmetic as follows (for two examples – identity and distributive). The method starts by looking at arithmetic examples, replaces the numbers with “any number”, and finishes with a letter representing a variable. The examples of identity and distributive are provided.

**Identity:**

\[7 + 0 = 7\]
\[23 + 0 = 23\]
\[\text{any number} + 0 = \text{any number}\]
\[x + 0 = x\]
\[8 \times 1 = 8\]
\[47 \times 1 = 47\] - and so on
\[\text{any number} \times 1 = \text{any number}\]
\[y \times 1 = y\]

**Distributive:**

\[3 \text{ tens} + 4 \text{ tens} = 7 \text{ tens}\]
\[3 \text{ eights} + 4 \text{ eights} = 7 \text{ eights} - \text{ and so on}\]
\[3 \text{ anythings} + 4 \text{ anythings} = 7 \text{ anythings}\]
\[3x + 4x = 7x\]
4. Equivalence and Equations

Equivalence and equations explores how to represent everyday life in terms of relationships. Thus, it studies the symbols, notation and rules for equations (number sentences with equal signs). In the long run, this is equations with numbers, operations and letters. To get to this point requires studying: (a) equations in arithmetic (no variables/letters); (b) equations with unknowns for which calculation is in arithmetic form (called pre-algebra by some curricula); and equations with variables where calculation is in algebraic form (considered to be full algebra).

It is imperative that students learn symbols as ways of telling stories about everyday life. Initially, the symbols will be with metric (e.g., numbers and operations symbols). However, as problems move to where not all numbers are provided (e.g., I bought a hat for $88 and a coat and spent $227 altogether), the symbols will include variables (pre-algebra and algebra). (See MaST activities in Section 7.)

![Sequence for teaching Equivalence and Equations](image)

The physical balance and length restrict the operations that can be used to addition, subtraction and simple multiplication. They are also kinaesthetic and time consuming – excellent for beginning the teaching. The diagrams of balances and number lines enable more activities to be completed but still have restrictions on operations. Thus, it is important that students understand that the balance and the lines must become abstract mathematical balances and lines and able to represent any operation (including division). This means combining the abstract balances/drawings with equations. This leads to the final step which is to dispense with materials and just use equations.
4.1 Major ideas and models

Major ideas

Equivalence and equations covers developing understanding of number sentences involving numbers, operations, variables, and equals, greater than and less than signs. These are called equations, inequations and expressions, and are defined as follows.

(a) An **equation** is a sentence, usually involving numbers, operations and variables, that has an equals (=) to show a relationship between two things (e.g., \(2 + 3 = 5\), \(2x + y = 16 = y2\)).

(b) An **inequation** is an equation with greater than or less than symbols (\(<\) or \(>\)) showing an order relationship (e.g., \(2 + 3 > 4\), \(2x + y < 16 + y2\)).

(c) An **expression** is one side of an equation; it has numbers, operators and variables but no equals or greater or less than symbols (e.g., \(2 + 3\), \(16 + y2\)). Thus an equation is the equivalence of two expressions and an inequation is an order relationship between two expressions. To study equations and inequations is also to study expressions.

The major ideas to be covered in equivalence and equations deal with using equations to model real life and manipulating and solving the equations to solve these real life problems. The sequence of activities designed to build this understanding and proficiency is as follows:

(a) introducing the notion of same and different and relating this to introduce equal, unequal, greater than and less than in length and mass (balance) situations;

(b) using mass and length in unnumbered situations to build understanding of equals and equations and to develop the equivalence and order principles;

(c) using mass and length in numbered situations to build understanding of arithmetic equations and reinforce the equivalence and order principles in numbered situations;

(d) relating arithmetic equations to real-life situations and vice versa (e.g., telling stories about the world);

(e) using mass and length models to introduce the balance principle that equations stay equal if the same thing is done to both sides of the equation;

(f) using mass and length models to introduce unknowns, and relate equations with unknowns to real-world situations and vice versa;

(g) extending mass and length models to mathematical versions (in picture form) where all operations are possible;

(h) using the balance principle to find solutions to equations with unknowns;

(i) developing rules/principles that enable expressions to be manipulated (including simplification and substitution); and

(j) introducing graphical representations of equations and showing how graphs, equations and everyday life relate.
Main models

The main models to be used are mass (the balance beam) and length (strips of papers and the double number line).

1. **Mass.** The materials/pictures begin with real balances (which can only really cover adding and some subtracting) and move onto pictures of “mathematical balances” (which can cover all operations) added with symbolic equations.

   Equals is shown by the balance being “in balance” and not equals by the balance being “out of balance” (see examples below).

   - **Equation:** $3 \times 4 + 2 = 12$
   - **Example:** $3 \times 4 + 2 \neq 16$

   The balance is good for developing the balance rule and, in picture form, handling all operations.

   - **Virtual balances** are also available online which provide a combination of the movement of the real balance with the range of operations available when using the picture balance.

2. **Length.** The materials for this model are strips of paper and single and double number lines (see below and the following page for diagrams).

   All these length models can handle addition and subtraction and the line and double number line can handle simple multiplication as well. However, more complex operations (particularly if they go into negative) are not possible. Please note that the number lines can be horizontal as here and vertical as later. The vertical line has the advantage of enabling an $=$ sign to be placed under it and a left and right hand side to be identified (as in an equation).
Length is also a useful material for teaching the balance principles, for example:

\[
\begin{align*}
A &= B + C \\
E &= B + D
\end{align*}
\]

**4.2 Very early Equivalence and Equation activities**

Equivalence and equations studies arithmetic (and algebra) as relationships. It builds understandings of equations and inequations, unknown and variable, algebraic equations, and the Balance principle. The major models on which teaching is based are balance (mass) and length models. Thus, the techniques used in Equivalence and Equations are similar to Arithmetic principles. Equivalence and equation activities can begin in Pre-Prep as the following sequence shows.

**Same and difference.**

**Step 1:** Students identify objects which are the same and which are different. They learn to describe what is the same and what is different about 2 objects. They sort objects into those that are the same and notice that different groups are different.

**Step 2:** This understanding of same and different is then considered in terms of, particularly, mass and length. For mass, teacher takes two plastic bags and places on students’ arms, puts things in the bags and allows children to feel when things are the same and when they are different (as below).

For length, look at objects (e.g., paper strips) and see if same or different lengths.
Step 3: Once same and different are understood, the material can be used to introduce the formal language for same and different, namely, equals, not equals, greater than, less than. After the formal language the symbols are introduced, namely =, ≠, <, and >. For example:

The technique is to discuss what is happening with respect to the balance [it is balanced] and introduce words and symbols by sticking the language and symbols on the balance. Relate the notion of balance to “equals” and imbalance to “not equals”. Note: Early on in the children’s primary years, similar techniques could be used to introduce “greater than” (>) and “less than” (<).

Note: length can also be used in this way but, maybe not as strongly. For example:

Step 4: Move hands along the balance to introduce equations for objects, e.g.
Unnumbered Activities

**Step 1:** The first formal activities should not use number; just different objects and different lengths. These are explored for equals and not equals and, later, greater than and less than. Children find different things to balance and not balance and record these as “equations” (not in the strictest sense), see example right:

**Step 2:** With direction, exploration with materials and objects can be used to find the equivalence principles (see below).

As pioneered by Davidov in Russia, unnumbered activities are an excellent way to begin work in a mathematics area. The lack of numbers appears to allow the students the freedom to explore structures and principles.

Similar work can be done with length and, because of ability to put things side by side, the length model can represent some things strongly, e.g.,
Numbered Activities

**Step 1:** Once unnumbered situations have been explored, numbers can be introduced by using same size weights (recommend small baked beans cans) and same size lengths (e.g. unifix cubes). Then we can use models to represent equations with numbers, e.g.,

![Equation Example](image)

It is important to read the equations from the materials, e.g., 2 cans plus 4 cans balances 6 cans so $2 + 4 = 6$, and reteach the principles, particularly symmetry, e.g.,

![Equation Example](image)

**Step 2:** Extend the models to inequations (e.g., greater than and less than), e.g.,

![Inequation Example](image)

**Step 3:** As students’ experience grows, extend models to introduce mathematical-balance pictures and double number lines which can handle more operations, e.g.,

![Mathematical-Balance Example](image)

**Note:** We have turned the cubes and the line vertical because it shows LHS and RHS. This does not have to be done but it makes the equation easier to relate to the picture.
Relating equations to real world situations

It is important to relate real world situations to equations. To do this, relate stories to actual components of the equation, for example, see on right:

A good way to teach the relation of symbols and stories is to use a material context. Two examples are, “The rice and the soap are the same as the pasta and the sugar”, and “the two weights and the four weights are the same as six weights”.

<table>
<thead>
<tr>
<th>Story</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two boys join three others</td>
<td>2 + 3</td>
</tr>
<tr>
<td>Two boys join three others, how many in all?</td>
<td>2 + 3 = 5</td>
</tr>
<tr>
<td>Two boys join three others to make 5 boys</td>
<td>2 + 3 = 5</td>
</tr>
<tr>
<td>Two boys join three others and this is the same number as 6 boys and one leaves</td>
<td>2 + 3 = 6 - 1</td>
</tr>
</tbody>
</table>

Rice + Soap = Pasta + sugar

2 + 4 = 6

It is important to reverse this process, for example, “what is a shopping story for 3+5=8?”

“I bought a chocolate for $3 and a hamburger for $5, and spent $8”

One can also use length processes, for example:

“There were 7 boys and 1 left, which was came to the same number as the girls, where 4 joined 2.”
4.3 Early to middle Equivalence and Equation activities

From the models developed in the early years, activities can now build the important principles for solving equations.

Balance principles activities

Balances can be used to explore what happens when extra weights (e.g., one weight) are added or removed from a balanced equation.

Students can be asked how to balance the equation again. There are 3 possibilities for the example above: (1) put the weight back again (this returns the equation to 2+3-5), (2) add another weight to the 3 on the LHS (this makes equation 1+4=5), and (3) remove a weight from RHS (this makes the equation 1+3=4). The third possibility is the beginning of the Balance principle and should be the focus of questioning (as on right).

Direct the students to add and remove different weights and to rebalance. With questioning, try to get students to generalise this process (e.g. “whatever you do to the one side you do to the other”) to the full Balance principle.

The Balance principle can also be introduced and demonstrated with length models (as on right).

The Balance principle can also be reinforced with mathematical-balance pictures and double number lines as follows.
Relating real world situations to equations

It is important to continue to reinforce the relationship between real world situations and equations. The relationship is best taught by experience in which equations are deconstructed into parts and related to stories and vice versa (i.e., stories are deconstructed and related to equations). Some examples are as follows.

**Story to equation:** “I bought 3 chocolates for $4 each and a pie for $6. I spent the same as June, who bought a meal for $14 and drinks for $4.”

\[
\begin{align*}
3 \times 4 + 6 &= 14 + 4 \\
2 \times ? + 6 &= 3 \times 8
\end{align*}
\]

**Equation to story:** “The equation is 2x+6 = 3x8; what story can this tell?”

“My dad and my mum gave me the same amount of money. I already had $6, so I was able to exactly pay for 3 meals at $8 each.”

One way to reinforce these relationships is with worksheets with headings as below in which teachers fill in one space in each row and the students fill in the other spaces. (Note: Students tend to have the most difficulty with creating their own stories.)

<table>
<thead>
<tr>
<th>Story</th>
<th>Number line</th>
<th>Balance</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I was given...</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. ....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3. ....</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4. ....</td>
<td></td>
<td></td>
<td>______ = ______</td>
</tr>
</tbody>
</table>

(Note: Students tend to have the most difficulty with creating their own stories.)
Introduce unknown and solve it

Discuss with students what they see an unknown as (students tend to choose “?”). Make a bag with the chosen symbol on it (e.g., ?), put in 3 weights and then balance it and 2 weights with 5 weights.

Discuss that the bag is unknown. Ask how we could find what is in the bag without opening it. Most students will know it is 3 because 3+2=5, but ask students to find a way to make it without this knowledge – state that the numbers could be large. Talk about how we can get the unknown on its own. Students can usually see that we can get unknown on its own by removing 2 weights and that this means removing 2 weights from RHS as below.

This can be translated to mathematical-balance diagrams and many examples done, e.g.,

The number line can also do this, first with blocks and then, more abstractly, with double number lines.

It is quite easy to extend this to more than one unknown and more than one type of unknown (see example on next page).
One type of unknown:

\[ ? + 7 = 2? + 3 \]

\[ 4 = ? \]

Two different types of unknowns:

\[ 2x + 3 = 15 \]

At this point, the process can be extended to symbolic equations. The balance drawings are the best way to make the transition because they can do all operations. Continue to stress that the drawings show a “mathematical balance” that can do all operations.

<table>
<thead>
<tr>
<th>Picture</th>
<th>? Notation</th>
<th>Variable Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ 2x + 3 ]</td>
<td>[ ? = 6 ]</td>
<td>[ x = 6 ]</td>
</tr>
</tbody>
</table>
Number line activities

At this point, it is useful to introduce single number line activities. In these activities, the line is used to act out real world situations. For example, “I had $20, I spent $8, got an extra $10 and then spent $15, how much do I have?” Moving along the line will give the answer.

It is possible to use this number line model when there is a variable. For example, “My dad gave me some money, I spent $12, I received $8, then my Mum gave me the same money as my Dad, how much do I have?” Need to come up with a symbol for the unknown – could be \( n \) or \( ? \). Then the line helps.

The line is useful for teaching inverse. This is particularly so for the inverse that makes an unknown on its own when solving an equation such as \( ? - 3 = 11 \). For example, how do we change \( ? - 3 \) so that we get back to \( ? \) on its own? The following line work can help students understand, and the line is a good model to explain or act out the process, e.g.,

Thus, the number line can be used to find the answer to unknowns. For example, “I went out and bought a CD for $23, this left me with $16, how much did I start with?” To solve this, make the start \( n \) and use the line as below; \( n \) goes to \( n - 23 \), this is $16, so +23 to get back to \( n \) which is $39.
4.4 Later equivalence and equation activities

In the upper primary and junior secondary grades, the skills learnt to model everyday activities and manipulate symbols can be widely used.

1. Modelling activities

One of the crucial things to develop is the ability to translate real world situations into equations and back again. This is done by teaching the students what the equation means as a story (Activity A) followed by activities interpreting the story in terms of symbols (Activity B – reversing A).

Activity A:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Story</th>
</tr>
</thead>
</table>
| 3 + 6 = 27 | • 3 means $3 \times$ any number; +6 means to add 6  
• So we need 3 lots of the same thing plus 6  
• What about maxi taxis?  
• “Three maxi taxi loads of children were brought to the game. 6 children were already there. This made 27 children.” |

Activity B:

<table>
<thead>
<tr>
<th>Equation</th>
<th>Story</th>
</tr>
</thead>
<tbody>
<tr>
<td>There are two times that are not given. They are the same. This is $t + t$ or $2t$. So, $2t + \text{other time} = 40$ Thus, $2t + 26 = 40$</td>
<td>“John waited for the lift, then spent 4 minutes travelling in the lift, then spent 15 minutes at his appointment and then waited the same amount of time as the first wait for the lift again. It was another 7 minutes before he left the building. He spent 40 minutes in the building.”</td>
</tr>
</tbody>
</table>

We can use the Balance principle on these equations to work out the unknowns, e.g.,

Activity A:

\[
\begin{align*}
3x + 6 &= 27 \\
3x &= 27 - 6 \\
x &= \frac{21}{3} = 7
\end{align*}
\]
Activity B:

2. Analysing the balance principle

To use the balance principle means to understand it and how it relates to other ideas. The following should be worked through with students.

Step 1: Discuss the balance principle. The balance principle states that if something is done to one side of an equation, the same has to be done to the other side to return to balance or to equal. Make sure students understand this. For example:

Step 2: Same does not mean same. Ensure students understand that adding the same thing to both sides does not have to be in the same form, for example, you could +5 to the left hand side and +7−2 to the right hand side.

Step 3: Relate equation and expression. Keeping expressions and equations the same uses opposite strategies (like addition and subtraction use opposite strategies in compensation). The inverse or equivalence principle for an expression states that if the expression is to stay the same value then +0 and ×1 are the only possibilities and any change has to be compensated by the inverse change, for example:

\[2x + 3 - 18 \neq 2x + 3 - 18 + y\] but \[2x + 3 + y = 2x + 3 + y - 18 - y\] (because \(+k-k = 0\))

The balance principle for equations and the equivalence principle for expressions, therefore, use the opposite strategy to ensure that any change has no effect. That is, to keep things the same, a change is repeated (on the other side) for an equation, while a change is undone or inversed (on the same side) for an expression.
Step 4: Solving for an unknown requires both expression and equation understandings: The problem of solving for unknowns is that both expression and equation actions have to be done together and they can be mixed up. For example, to solve $2y + 3 = 35$ for $y$ we need to look first at the expression and then the equation, as follows:

**Expression:**
- $2y + 3$ → have to get $y$ alone
- $y$ has been $\times 2$ and $+3$
- so $-3$ and $\div 2$ will get $y$ alone

**Equation:**
- $2y + 3 = 35$ → to keep balance $-3$ and $\div 2$ on RHS
- $-3$: $2y + 3 - 3 = 35 - 3$
- $\div 2$: $2y/2 = 32/2$
- $y = 16$

However, some students get this confused and either invert both or do the same for both; and some students get it doubly wrong but end up with the correct answer, for example:

$2y + 3 = 35$ → have to $\times 2$ and $+3$ to get $y$
→ equation means have to do opposite to RHS
→ so, $y = 35 - 3 + 2 = 16$

**Step 4:** Spend time discussing the two principles. Ensure students understand the distinction. If there are problems, return the students to the models as in the example that follows.

**Example:** Consider $2x + 3 = 39$. The first stage is to consider the expression $2x + 3$ and use inverses to get $x$ alone in the expression. This can be done with number line as follows.

![Number Line Diagram](image)

Then, the Balance principle can be modelled on a balance as follows, first subtracting 3 and then dividing by 2 as the number line indicates (as shown on next page).

![Balance Principle Diagram](image)

**Note:** It is also necessary to realise that if you start with an expression and divide and add, then the inverse is the **inverse of each operation and the inverse of their order**, for example, for this change:

![Order of Operations](image)

The inverse change is:

![Inverse Operations](image)
3. **Substitution and simplification**

Once we are familiar with equations with variables, we can use these independently of real world situations and the possible quantities the variables may represent. Other than solving for unknowns, the two important mathematical ideas are substitution and simplification.

**Substitution:** Substitution involves replacing the variable with a number and using arithmetic to work out the answer, for example:

\[
2x+5 \text{ when } x = 7: \quad 2x+5 = 2\times7+5 = 19
\]
\[
3x+y-7 \text{ when } x = 5, y = 9: \quad 3x+y-7 = 3\times5+9-7 = 17
\]

**Simplification:** Simplification involves using arithmetic understandings to calculate with variables. An effective way to build understanding of the rules of variable or algebraic calculation is to build the calculation rules from patterns seen in arithmetic (see below). However, it is important not to think that letters stand for objects, they stand for numbers of objects. For example, if we have a box of apples and we use the letter a, it stands not for apples but the number of apples in the box. Thus, patterns for letters must come from numbers.

- 3 apples + 2 apples = 5 apples
- 3 eights + 2 eights = 5 eights
- 3 hundreds + 2 hundreds = 5 hundreds
- 3 any number + 2 same number = 5 of any number
- \(3x + 2x = 5x\)

This method of finding the pattern from arithmetic can be repeated to show simplifications such as \(3x + 2y + 4x = 7x + 2y, 3x = x + x + x, 5 \times 3m = 15m,\) multiplication of \(x\) by itself = \(x^2\) and \(3a \times 4b = 12ab.\)

4. **Using sequences and materials**

To teach algebraic manipulation of expressions, including substitutions and simplifications, it is useful to follow the sequences below:

**Sequence 1 – Complex arithmetic activities as a step between arithmetic and algebra:**

The difference between arithmetic and algebra is that arithmetic expressions have separate processes, e.g., \(2 \times 3 + 4,\) and products (the answer), e.g., \(10,\) while algebraic expressions have processes and products, e.g., \(2x+4,\) as the same thing. Expressions can be closed (calculated out), e.g., \(2\times3+4\) is \(6+4\) is \(10,\) in arithmetic but not in algebra. This means that, for most students, arithmetic expressions are always simple – a sequence of binary calculations to an answer, e.g., \(2\times3+4=6+4=10.\) However, algebra is mostly complex - consisting of expressions of 2 or more operations which cannot be calculated. This means that expressions have to be understood as processes involving many operations in algebra, but can be understood as a series of one-operation products (answers) in arithmetic.

Thus, we should not teach the traditional sequence from arithmetic to mathematics which is really a large jump from arithmetic with one operation (which is understood as answers) to algebra with more than one operation (which is understood as processes), as in the first diagram below.
We should teach it has two paths as in the diagram below (where we spend time with simple algebra and complex arithmetic). By complex arithmetic, we mean understanding $2\times3+4$ as a process and not as $2\times3$ and $6+4$.

<table>
<thead>
<tr>
<th>Path 1</th>
<th>Path 2</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Binary Arithmetic</strong></td>
<td><strong>Complex Arithmetic</strong></td>
</tr>
<tr>
<td>(e.g. $2 \times 5$, $5 + 3$)</td>
<td>(e.g. $2 \times 5 - 6$, $5 + 3 - 4$)</td>
</tr>
<tr>
<td><strong>Binary Algebra</strong></td>
<td><strong>Complex Algebra</strong></td>
</tr>
<tr>
<td>(e.g. $2x$, $x + 3$)</td>
<td>(e.g. $2x - 6x + 3 - 4$)</td>
</tr>
</tbody>
</table>

It is difficult to think of complex arithmetic examples because they rely on students not closing on the first binary part. An example of a good activity is:

“Work out $\frac{24 + 36}{6}$ without calculating $24 + 36$?”

**Sequence 2 – Pre-algebra activities as a step from arithmetic to algebra:**

The first uses of letters are as unknowns. When used as unknowns, computation is predominantly arithmetic. This is called the pre-algebra stage. It is useful to go through this stage as in the diagram below.

<table>
<thead>
<tr>
<th>Arithmetic</th>
<th>Pre-algebra</th>
<th>Algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>e.g. $2 + 3 = 5$</td>
<td>$x + 5 = 16$</td>
<td>e.g. $x + 3 = 2x + 1$</td>
</tr>
<tr>
<td>$35$</td>
<td>$3x + 7 = 24$</td>
<td>$x + 3y + 4x - 2y = 15$</td>
</tr>
<tr>
<td>$7 + 8 = 13$</td>
<td>$3x$</td>
<td>$3x$</td>
</tr>
</tbody>
</table>

In developing the algebra stage of sequence 2, we need, according to sequence 1, to build understanding of $x+2$ and $3x$, and extend this to more complex examples. One way to do this is to use physical materials to model the algebraic expressions. Some materials and their uses are below.
Possible physical materials:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Numbers</th>
<th>Variable squared</th>
</tr>
</thead>
<tbody>
<tr>
<td>Envelope or box</td>
<td>Counters</td>
<td></td>
</tr>
<tr>
<td>Cups</td>
<td>Counters</td>
<td></td>
</tr>
<tr>
<td>Fixed length</td>
<td>A shorter length</td>
<td>A square tile</td>
</tr>
</tbody>
</table>

Uses:

\[3x + 2\]  \[\text{3 cups and 2 counters}\]
\[3(x + 2)\]  \[\text{3 lots of 1 cup and 2 counters}\]

\[(2x + 1)(x + 3) = 2x^2 + 7x + 3\]

Thus, with materials, one can state a real world situation, model it and give language and symbols. An activity to reinforce this is to get students to fill in the 4 columns of a table such as that below.

<table>
<thead>
<tr>
<th>Story</th>
<th>Material</th>
<th>Language</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. I bought 4 pies and a $7 cake.</td>
<td>[\text{4 unknowns + 7 dollars}]</td>
<td>[4p + 7 dollars]</td>
<td></td>
</tr>
</tbody>
</table>

It is also important that this situation is reversed, i.e. start with symbols, state language, model with materials and create a story. The creation of stories is powerful in teaching how to interpret stories.
5. Change and Functions

Change and functions explore how to represent everyday life in terms of change or transformation. Thus, it studies the symbols, notation and rules for change and functions (including tables, Arrowmath symbols, and graphs). In the long run, it returns to representing functions using equation notation. To get to this point requires studying: (a) change in unnumbered situations; (b) change in arithmetic situations (numbers and operations); and (c) change in algebraic situations (numbers, operations and variables). The teaching sequence is in Figure 7.

![Figure 7. Sequence for teaching Change and Functions](image)

It is important that Arrowmath and equations be related at the end so that both relationship and change ideas can be applied to algebraic situations using the same notational forms (the expression and the equation).

5.1 Major Ideas and models

**Major ideas**

The major ideas to be covered in change and functions deal with mathematical forms that describe change (such as functions) and with the major ideas that emerge from changes, namely,

(a) changes that do not change anything (e.g., +0, ×1, a 360° turn) – the identity principle; and

(b) changes that reverse other changes (e.g., −6 reversing +6, ÷8 reversing ×8) – the inverse principle and backtracking.

Real-world situations can be translated into relationships or changes. For example, “2 joining 3 to make 5” is a relationship if considered as 2+3 = 5 and is a change if considered as on right.
Similarly, the two triangles below can be considered as a similarity relationship or a change due to projection that enlarges the first shape to the second shape.

![Triangles diagram]

Relationships are most often represented as equations and this form of notation is good for seeing equals as balance and for applying the balance principal (that there is a LHS and a RHS and they have to stay in balance).

Changes can also be represented as equations but it is easier to understand them if they are represented by Arrowmath notation. For example, the situation, I bought some $3 pies and a $5 chocolate, how much did I spend? cannot be calculated because there is not enough information given. However, if we knew the number of pies, we could calculate the answer by multiplying this number by 3 and adding 5. Thus, the notation can be thought of as the equation \( n \times 3 + 5 \) (or \( 3n + 5 \)), or in arrows as on right:

![Arrows diagram]

The Arrowmath notation makes studying the change easy. First, changing forward, it is easy to work out what money will be paid for differing numbers of pies, for example, as below.

Thus, if the number of pies is 7, then the answer is $26.

Second, by reversing the change, it is possible to find the number of pies if I paid $38 for the pies and chocolate (see on right). We use the inverses of the operations and backtrack (as shown in the bottom arrows) to the answer of 11 pies.

Thus, the major ideas that can be developed in this strand relate to inverse but include the following:

(a) developing the notion of change and inverse (backtracking) in unnumbered situations;
(b) extending the notions of change and inverse to numbers and operations (first with addition and subtraction, second with multiplication and division, and third with more than one operation);
(c) introducing drawings (number lines and function machines), tables and Arrowmath notation to describe changes and inverses;
(d) relating change and inverse (backtracking) to real-world situations and vice versa;
(e) generalising change and inverse and using this to introduce variables and algebraic expressions and equations (including conversions between Arrowmath and equation notations);
(f) interpreting real-world problems in terms of change and using backtracking to solve for unknowns;
(g) representing generalised change with graphs and relating real-world situations, Arrowmath and equation notation, and graphs and change to graphs in all directions.
Main Models

There are two major models that can help with change, namely:

1. Number lines. As we saw in equivalence and equations, operations can be represented on number lines by arrows, for example, the problem with the pies and chocolate \((3n+5 = n+n+n+5 = 38)\) can be represented on a double number line as in the example on the right.

   The number of pies \((n)\) can be worked out by first crossing out the 5 which gives \(3n = n+n+n = 33\) and then sharing the 33 evenly between the three \(n\)s as on right. It gives the answer of 11 pies.

   However, this reflects the balance approach from equivalence and equations and does not use backtracking (inverse).

   There is another way to represent the problem on a number line as change. This way shows operations as changes on the line as shown below.

   ![Number Line Example]

   Backtracking is going back along the line and undoing what has been changed as seen below.

   ![Backtracking Example]

2. Function Machines. This is the major model. A “machine” is constructed. It can be a whiteboard, or blackboard or a box with holes in it and takeout cards, for example:

   ![Function Machine Example]

   Function machines operate as follows.

   (a) Situations are described – “I sold small prints for $20 each, I paid $200 to rent the site, how much money did I make?”

   (b) Situations are translated to activities on an unknown or variable, e.g., \(P\) is the number of prints so:
(c) These are translated to two function machines and the changes are acted out by students with numbers on cards.

(d) Examples are put in an input/output table starting with input numbers (shown right) and students act out the change.

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>160</td>
<td>-40</td>
</tr>
<tr>
<td>10</td>
<td>200</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>400</td>
<td>200</td>
</tr>
</tbody>
</table>

(e) Numbers are then put into the middle or output and students act out how to find the other sections of the table.

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>?</td>
<td>240</td>
<td>?</td>
</tr>
<tr>
<td>?</td>
<td>?</td>
<td>100</td>
</tr>
</tbody>
</table>

(f) A variable amount, say n, is considered and discussed at the function machines and responses included.

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>20n</td>
<td>20n+5</td>
</tr>
</tbody>
</table>

(g) Then different examples (including variables) are given at output and students backtrack to find input using the function machines and recording on input/output table.

The changes are reversed, and the inverse operations and inverse sequence is found and written in Arrovmath notation.

(h) Then the change and inverse change are written as equations (it takes time to get used to how these two are connected),

(i) The change is then graphed as on right.

Everything is reversed – a graph is drawn and students have to find equations (forward and inverse), change equations, and Arrovmath notation, put examples on the input/output table, and construct the function machines.

Notes: (a) the 11 steps above are the final endpoint of teaching across PP-9, only a few of these steps would be undertaken for young children, and (b) if this approach is learnt, it makes such things as the inverse of a function easy.

For example, a function f(x) = 2x+1 can be considered in Arrovmath notation as on right. Thus, the inverse function, g(x), is this Arrovmath notation in reverse (backtracked) as on right. Changing this backtracking to an equation, this means that the inverse function is g(x) = (x-1) ÷ 2.
5.2 Very early Change and Functions activities

Change and function covers operations as change and is designed as a precursor to functions. The sequence for teaching is to move from unnumbered to numbered activities, addition and subtraction to multiplication and division, and one operation to multiple operations. It introduces input/output tables and Arrowmath notation. It develops inverse of one operation, inverse of sequence of operations, and backtracking to solve for unknowns. It relates Arrowmath notation to equations, and graphs the results, relating all aspects in all directions. It continually relates symbols and models back to everyday situations so that it can model reality. Finally, it allows change to be generalised into a rule using algebraic notation, thus introducing variable and leading to function.

The early years are when unnumbered activities are used to develop the language of change (the words change, input and output) and the notion of what change and reversing change are.

Introduce notion of change

Discuss with students how things change (e.g. clean things, grow things, weather gets hotter, things are moved around, we change clothing, change hair colour, and so on). Discuss before and after – before washing hair, after washing hair, before putting on a dress, after putting on a dress. Take photographs – use these as before and after discussion points, e.g., what happened here?). Look at relationship patterns – what goes with what? For example, shoes with feet, shirts on bodies, hats on heads, and so on. Focus on how before and after have to be related in some way.

Play games like “switch” (or its commercial form, UNO). If a spade is put down, have to follow it. Can change it you can put the same value on top, and so on. Play snakes and ladders – things change if you land on a snake or a ladder.

Unnumbered change activities

Set up a function machine that will have an input and output and a rule for change, see example on right for whiteboard.

Give students a small copy of the board to record their changes (doing this with a small chalk board was very successful in one school).

Choose an unnumbered change and put this in the RULE box. Discuss what input and output are – act out some changes. Have children walk in on the left with a picture of a thing to be changed and stick this on input. Discuss what it could change to. Give students a picture of this change to put on RHS. Students should also record on their recording sheets or chalk boards, the input and output (the in and the out).

Examples of unnumbered change include:

(a) lower case to capital letter (e.g. input h and output H);
(b) “cook it” (e.g. input picture of potato and output a picture of chips);
(c) “wash it” (e.g. dirty car to clean car);
(d) “wear it” (e.g. hand to glove, foot to shoe);
(e) add “at” (e.g. b to bat, fl to flat);
(f) add “ing” (e.g. r to ring, s to sing); and
(g) move first letter of word to end of word.
Involv[e]e students in bringing out picture cards (or potatoes or letters or whatever is relevant for the RULE) and working out what the output will be. Get students to discuss what is happening, and encourage students to think of things to change and even to think of changes.

**Note:** Attribute logic blocks or pattern blocks can also be used – changes can be blue to red or large to small or triangle to square. The problem here is that some things do not change.

**Inverse of Unnumbered Activities:**

Use the function machine from (2) above to look at the notion of inverse. Make up a matching set of pictures before and after cooking (e.g., pasta in a packet to spaghetti bolognaise, and so on). Organise children to come out front and pick an input card. Get children to show the picture to the class and stick it on the input side of the table. Discuss what would be on the output side. Select the likely picture from output cards.

After doing this for some time, ask students to select an output card and stand on the RHS. Discuss what the input card could be. Repeat this as often as required. Initially, get students to think what input could give this output? i.e., we have mashed potato, what could we cook to get this? Progress to “thinking backwards”, e.g., what do we get when we “uncook” the mashed potato?

This is easier done with, for example, the “add ‘at’” rule. Here, input of s goes to output of sat – “add” the “at.” When we look at an output of rat, it is fairly straightforward to consider removing, or “taking away,” the “at” to find an input of “r”. Similarly, input and output makes “h” into “H” if “capitalise” is the change, and we can think of output to input as “uncapitalising”, e.g., “R” to “r”.

**Extending Change:**

There are two ways we can begin extending what we have done in the above.

1. Consider two changes – one after the other (e.g. 2 function machines together), for example:

   ![Function Machine Diagram]

   Now children can move through 2 changes – 1st change b to B and 2nd change from B to Bat. Can also try to reverse both changes, e.g., if we ended with Fat, then this goes back to F and then f. (It should be noted that changes like e to Eat could be challenging as well as Ef to Flat.)

2. Consider bringing in number. This could be done initially by adding two extra counters or removing 3 counters from plastic bags with counters, or using input and output cards showing sets of counters, as for function machine on right.

   For this function machine, an input of 4 counters would give an output of 6 counters, while an output of 9 counters comes from an input of 7 counters.
5.3 Early to middle Change and Function activities

In the middle years Change and Functions work progresses to include numbered activities (similar to the progression within equivalence and equations in section 3.3) – first with the operations of addition and subtraction and then with multiplication and division.

One operation – addition and subtraction

Move onto operations with formal symbols. Set up a function machine that adds or subtracts a small number.

The whiteboard function machine is still excellent but for this we will move onto the “robot” function machine. Basically it is a large box (in which students can stand) with a small “head”, two openings each side, a rule hung around the neck or from the top (if there is no “head”) as in example on right.

Two card sets are made – numbers 1 to 20 for Input cards and 1 to 30 for Output cards. Students, in twos, bring an Input card to LHS of robot and place it in the opening. Students inside add 3 and push Output card out RHS opening. Remaining students have a calculator to check that correct change has been made (e.g., 6 to 9) and a worksheet on which to record Input and Output numbers.

The following is a sequence of activities found useful.

1. Give students a real world problem that adds/subtracts a small number. E.g. “It costs $5 to have a present wrapped. What is the total cost of present and wrapping?” Discuss what we can do with this. [We cannot get answer as is but there are 2 things that can be done: (i) if given present’s price, can work out total cost, e.g., present if $36, total cost $41; and (ii) if given total cost, can work out present cost, e.g., total cost $24, present $18.]

2. Students consider problem as a change – ask “what is the operation” and then draw a function machine as on right.

3. Act out change with the function machine. Organise a student to go into robot with Output cards. Give other students Input card numbers and ask them to bring them out front, in turn, to Input and then collect a changed card at Output.

4. Fill in Input-Output table. Students should follow the function machine activity with a calculator, checking calculations and filling in Input–Output tables. Ask students to complete tables without watching a student at the front use the function machine. Reverse the change. Teacher directs a student to collect an Output card without showing Input. Ask class what was the Input card. Walk the student backwards from Output to Input as you are doing this. Discuss options and how to find this inverse number. Teacher provides a series of Input and Output numbers for students to fill in on their Input – Output tables. Have large numbers as part of this.

5. Develop inverse. Teacher leads discussion on quick ways to find the inverses and encourages students to see that –5 gives inverse or +5.

6. Use Arrowsmath notation. Students are directed to write both changes as Arrowsmath diagrams using examples (as on right).

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7. Generalise the change and its reverse. Choose a student and ask to go to Input. Tell other students that this student has a number to Input in but does not know what it is. Get class to discuss what Output would be. Do the same for any Output number – what would be the input? Give students a variety of large numbers to say what the change would be. Get students to write their rules in language. Repeat this for the inverse change. Move onto symbols (but do not push for accuracy or for everyone getting the answer). Ask what the Output would be if Input was n? Ask what the Input would be if Output was k? See if students can write \( n + 5 \) and \( k - 5 \).

8. Reverse everything. Give students a generalisation of a change (can give it as language or as examples of numbers). Ask the students to represent the change and its inverse, with examples, using Arrowmath notation, fill in an Input-Output table for some values, draw the change as a function machine, and create a problem for it.

For example:

\[
\begin{array}{c|c}
\text{Input } n & n - 3 \text{ Output} \\
\hline
6 & 3 \\
11 & 8 \\
\hline
\text{Change} & -3 \\
\text{Inverse change} & +3
\end{array}
\]

**Arrowmath activities.**

It is useful for students to think of operations as change. So build the idea of arithmetic excursions, travelling from number to number by actions or operators, as below.

\[2 \rightarrow 7 \rightarrow 14 \rightarrow 40 \rightarrow 20 \rightarrow 85\]

Some ideas for activities, called “Arrowmath excursions”, include using calculators to:

(a) go from 2 to 62 by 4 changes;
(b) go from 68 to 1001 by 7 changes;
(c) go from 687 to 23 going through 2099 on the way;
(d) make a long journey from 3679 to 9763 passing through 2 000 001 on the way; and
(e) go from 654 to 268 by multiplications only.

Encourage students to use calculators without first working out in their head where they are going. The aim is for them to understand that making a number larger is achieved through adding, multiplying or dividing by a fraction, similarly reducing is subtracting, dividing or multiplying by a fraction. If get a decimal, just subtract it on the next move.

Use Arrowmath notation to study properties for a variety of numbers. Some examples of activities are below. (Note – these examples use ? as “any number”.)

1. Does the change on the left give the same Output for given Input as the change on the right?

\[
\begin{array}{c|c}
\text{Input} & 4 \text{ Output} \\
\hline
? & ? \\
? & ? \\
\hline
\text{Change} & x4 \\
\text{Inverse change} & x3
\end{array}
\]

2. Does the change on the left give the same Output for given Input as the change on the right? Relate this answer to the previous example – what does this mean?

\[
\begin{array}{c|c}
\text{Input} & 3 \text{ Output} \\
\hline
? & ? \\
? & ? \\
\hline
\text{Change} & +4 \\
\text{Inverse change} & -4
\end{array}
\]
3. Reverse Arrowmath excursions and use Arrowmath to study inverse, as on right.

Importantly, there is a need at the end of middle school to begin relating Arrowmath notation to equations, as below.

Use the inverse of change and relation to equations to make up “talking calculator” activities. Discover which numbers upside down form letters (e.g. 0 is o, 1 is l, 2 is s, and so on). Make up a number which upside down is a word (like “shells”). Take this number, make changes to it with an Arrowmath excursion and follow these with calculator. Reverse the sequence of changes and write as an equation – it should equal the original number but leave this place blank (i.e., do not show the original number as the answer to the reverse calculation). Make this into a worksheet as below. Other students take worksheet, do the calculation on a calculator, turn the calculator upside down and arrive at the answer. Google “talking calculator” for more examples.
5.4 Later Change and Function activities

One operation – multiplication and division.

This repeats the activity from “One operation – addition and subtraction” (p.55). The function machine is set up for changes like ×5 and ÷4. The Input and Output cards have to be specifically selected. For change ÷4, Input cards are 4, 8, 12, 16 and so on to 80, while Output cards are 1 to 20. As for addition and subtraction, a student is put in the robot function machine while other students bring up Input cards and receive Output cards, and later bring up Output cards in order to work out the inverse that gives the Input. The idea is to cover steps as below (using example of ÷4):

(a) starting with a real world problem;
(b) drawing and setting up a function machine for this problem;
(c) filling in an Input–Output table;
(d) drawing Arrowmath diagrams (forward and reverse);
(e) conceptualising inverse as ×4;
(f) generalise change in language and using variables, e.g., \( n÷4 \) for Input \( n \), and \( k×4 \) for Output \( k \); and
(g) reversing the whole process – go from, say, \( n × 3 \) right through to real world problem.

Two function machines – all operations.

This repeats the activity from “(5) One operation – addition and subtraction” but with two function machines, for example:

This means that there are three columns in the Input–Output Table and three generalisations, see above right for example (circled item is starting point).

Inverse is important here, as is the Arrowmath, as it shows that, as well as inverting all operations, the order of the operations is also reversed, e.g.,

\[
\begin{align*}
6 & \quad 18 \quad 22 \\
1st & \quad 2nd & \\
\times 3 & \quad + 4 & \\
\end{align*}
\]

\[
\begin{align*}
11 & \quad 33 \quad 37 \\
2nd & \quad 1st & \\
\leftarrow & \quad \leftarrow & \\
\quad 3 & \quad - 4 & \\
\end{align*}
\]

Once again it is important to go both ways: (a) from real world problems to drawing to table to Arrowmath diagrams to generalisation; and (b) then reverse from a generalisation to Arrowmath to table to drawing to real world problem. Do not be tempted to miss the real world problem; it is crucial to relate the function machine to everyday life.

Note: For Middle years, do not expect or require students to successfully generalise for a variable \( n \). Remember from Growing Patterns (section 3.1) that students go through the following stages in generalising:

1. Not being able to generalise
2. Quasi-generalisation – doing it for any number given
3. Saying it in language
4. Writing it with letters

Note: For Middle years, do not expect or require students to successfully generalise for a variable \( n \). Remember from Growing Patterns (section 3.1) that students go through the following stages in generalising:

- Not being able to generalise
- Quasi-generalisation – doing it for any number given
- Saying it in language
- Writing it with letters
Later change and functions activities

In the final years of PP-9, change and function activities are used to generalise, introduce variable and algebraic expressions, draw graphs, and introduce functions.

1. Two operations and backtracking

This activity repeats and extends the activities in (5), (7) and (8) (p. 55-56) by formalising the process of backtracking. The steps are as follows:


Step 2: Discuss what can be done with this problem: encourage students to realise that if they know how many students, they can work out how much spent; and if they know how much was spent, they can work out how many students. Note that these things can be done are expressed as changes: forward – students to spending; and backward – spending to students.

Step 3: Work out operations used and construct/draw function machines, e.g., robots as below.

![Function machines diagram]

Step 4: Do examples and record on Input – Middle – Output tables. Students in robots with cards act out what happens. Other students check with calculators as well as record on tables. Note that in the table on right, the circled numbers are given and the rest have to be worked out and filled in.

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>15</td>
<td>22</td>
</tr>
<tr>
<td>9</td>
<td>27</td>
<td>34</td>
</tr>
<tr>
<td>18</td>
<td>54</td>
<td>61</td>
</tr>
</tbody>
</table>

Step 5: Look at reversing process. Again act this out, check with calculators and record on Input –Middle – Output tables. Again, the circled numbers are given and the rest are worked out and filled in by students.

Discuss with students until they see that reversing involves the inverse of the operators, e.g., ×3 goes to ÷3 and +7 goes to –7. Also ensure students see that order of operations is reversed, e.g., ×3+7 goes to –7÷3. Get students to stand on Output side with say 22 on a card and walk them backwards to the Middle and the Input side showing the inverses as the students walk backwards. Introduce the term “backtracking”.

Step 6: Record forward and backward (reverse) as Arrowmath, e.g.,

![Arrowmath example]

Step 7: Generalise the forward and backward changes by using letters and requiring students to complete Input–Middle-Output tables as on right. Again, the circled letters are starting points – the rest are worked out:

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>n</td>
<td>3n</td>
<td>3n + 7</td>
</tr>
<tr>
<td>p</td>
<td>3p</td>
<td>3p + 7</td>
</tr>
<tr>
<td>k</td>
<td>k – 7</td>
<td>k</td>
</tr>
</tbody>
</table>

Step 8: Reverse everything – start with a generalisation, e.g., 2n + 3, and work through Arrowmath, chart and function machine to a real world problems
Step 9: Use backtracking to solve real world problems as follows.

(a) Start with problem – “Each team member at 3 litres of water and the truck carried 25 litres. How many in team if there was 58 litres of water to be bought?”

(b) Put this into an Arrowmath. $\begin{array}{c} \text{?} \times 3 \rightarrow + 25 \\ 58 \end{array}$

(c) Reverse this (backtrack) to show there are 11 members $\begin{array}{c} 11 \xleftarrow{3} 33 \xleftarrow{25} 58 \end{array}$

(d) Use a worksheet where one of the 4 columns is filled in and the rest are to be completed.

<table>
<thead>
<tr>
<th>Real World problems</th>
<th>Forwards Arrowmaths</th>
<th>Backwards Arrowmaths</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Linear equations and backtracking

The first stage here is to relate real world problems to equations.

Step 1: Begin with Arrowmath excursions, e.g., $6 \times 5 \rightarrow 16 \rightarrow 14$

Step 2: Write these as equations and discuss relationship between Arrowmath and equations. $6 \times 5 - 16 = 14$

Step 3: Introduce letters into Arrowmath, e.g., $p \times 4 \rightarrow 88$

Step 4: Write these as equations. $(p + 6) \times 4 = 88$

Step 5: Solve these by backtracking (gives $p = 88 \div 4 - 6 = 22 - 6 = 16$) $\begin{array}{c} p \xleftarrow{-6} \xleftarrow{+4} 88 \end{array}$

The second stage is to translate real world problems to equations and solve by thinking backtracking, e.g.,

Step 1: Give problem “4 teams of players plus 9 adults got on buses. There were 57 people on the buses. How many in each team?”

Step 2: Translate to Arrowmath and equation (have to identify unknown and give it a letter), e.g., $\begin{array}{c} ? \times 4 \rightarrow + 9 \\ 57 \end{array}$

Step 3: Backtrack to get $t$, e.g., $t \times 4 + 9 = 57$ or $4t + 9 = 57$

$t \xleftarrow{+4} \xleftarrow{-9} 57$

The third stage is to go from equations to answers by thinking backtracking, e.g.,

Equation: $2x + 5 = 17$
Thinking: $x \times 2 \rightarrow + 5 \rightarrow 17$

$\begin{array}{c} x = 17 - 5 \\ 2 = 6 \end{array}$

However, it is always important to relate to real world situations.
3. Function machines and graphing

This extends Step 9 (p. 60) to graphing. The steps are as follows.

**Step 1:** Start with a problem “Five fishermen caught the limit of fish. They gave 7 fish away. This left them with 33 fish. What was the limit?”

**Step 2:** Translate to function machine (i.e., translate problem to changes) and identify change operations.

![Function Machine Diagram]

**Step 3:** Complete Input–Middle–Output table, identify reverse (inverses), and write change as Arrowmath and equations.

\[
\begin{array}{ccc}
? & x 5 & - 7 \\
33 & 8 & + 7 \\
5 \cdot 7 = 33 & ? = (33 + 7) / 4 = 8
\end{array}
\]

**Step 4:** Generalise forward and backward change, e.g., Input \(n\) gives Output \(5n - 7\). Output \(k\) gives Input ——

**Step 5:** Fill in an Input-Middle-Output table. Use Table points to plot a graph.

<table>
<thead>
<tr>
<th>Input</th>
<th>Middle</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>-2</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>15</td>
<td>8</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>13</td>
</tr>
</tbody>
</table>

**Step 6:** Reverse by going from graph to problem.
4. Linear functions and reverse linear functions

Functions can be considered in this “change” way by the following steps.

**Step 1:** Look at change from \( x \) to \( y \), e.g.,

\[
\begin{align*}
  x & \quad \rightarrow \quad 4 \quad \rightarrow \quad 7 \\
  y & \quad \rightarrow
\end{align*}
\]

This can be rewritten as an equation, e.g., \( 4x + 7 = y \) or \( y = 4x + 7 \)

**Step 2:** Introduce **new notation** which represents function as \( f \) with \( f(x) \) to denote variable being used.

**Step 3:** Get students to think of **functions as change**, e.g.,

\[
\begin{align*}
  f(x) = 4x - 7 \text{ is same as } & \quad x \quad \rightarrow \quad 4 \quad \rightarrow \quad -7 \\
  f(x) \quad \text{or} \quad y & \quad \rightarrow \quad f(x) \quad \text{or} \quad y
\end{align*}
\]

**Step 4:** This leads to **tables and graphs** as below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) ) or ( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-7</td>
</tr>
<tr>
<td>1</td>
<td>-3</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

**Step 5:** Easy to find **inverse function**, just use backtracking:

Thus if stay with \( x \) as variable for all functions and introduce notation of inverse function as \( g \), we have inverse function for \( f(x) = 4x - 7 \) being:

\[
\begin{align*}
  g(x) = & \quad \frac{x + 7}{4} \\
  \text{Reverse} & \quad \text{Function} = \quad \frac{y + 7}{4}
\end{align*}
\]

**Step 6:** **Reverse everything** by starting from function or graph and using these to construct Arrowmath notation for change and reverse change, Input-Output table, drawing of function machine, and story (real world situation.)
6. Cultural implications

This section of the Algebra booklet discusses (a) the relationship between abstraction and algebra, (b) how abstraction and structure are related to holistic teaching, (c) how Aboriginal, Torres Strait Islander and low SES learning can be facilitated by holistic teaching, and (d) implications for teaching Patterns and Algebra to Aboriginal and Torres Strait Islander and low SES students.

6.1 Algebra as abstraction of abstraction

Abstraction is a process by which a generality is determined from particular examples. In Western mathematics, an important abstraction is number. By experiencing, for example, many examples of two items (e.g., 2 eyes, 2 hands, 2 chairs, 2 children, and so on), learners generalise the language “two” and the symbol “2” as representing the “twoness” that is common to the examples. In a similar way, learners gradually build understanding of the language and symbols of all numbers.

When numbers and their names and symbols are new to learners, meaning lies with the items. For example, $2+3$ is thought of as 2 items and 3 items. Counting all the items gives the solution 5 items. Thus $2+3 = 5$ is thought of as 2 items joining 3 items to make 5 items. The focus of thinking is on the items. However, over time as more and more experience is gained, it becomes less necessary to think of items when we use numbers. After a while, $2+3$ can be considered as equal to 5 without having to think of 2, 3 and 5 as specific items. The thinking simply happens on the symbols 2, 3 and 5. That is, the numbers become the focus or “objects” of thought; not the items that underlie them. At this point, the activity with real world items has been abstracted to numbers and arithmetic.

However, abstraction does not stop with number. After a further time, learners start to see that sometimes things are the same regardless of the size and type of the numbers. An example of this are “turn arounds” (what is mathematically called the commutative principle), that is, for any number addition is the same regardless of the order in which numbers are added (e.g., $2+3 = 3+2$; $656+172 = 172+656$; $3\frac{1}{4} + 2\frac{1}{5} = 2\frac{1}{5} + 3\frac{1}{4}$, and so on). For this principle, letters such as $x$ and $y$ can be introduced as symbols for variables (i.e., to stand for “any number”) and used to represent the principle, that is, $x+y = y+x$. (Note: The commutative principle can be extended to more than two numbers and to algebra, and it only holds for addition and multiplication.)

Similar to numbers, when variables and their names and symbols (letters) are new to learners, meaning lies with the numbers that the variables could represent. For example, $2x+3$ is thought of as two multiplied by “any number” plus 3. Solving $2x+3 = 11$ means thinking like “I have a number, I multiply it by two, add 3 and end up at 11; to solve it, I subtract the 3 from 11 (get 8) and divide the 8 by 2 (get 4), so $x = 4$”. The focus of thinking is on the numbers. However, over time as more experience is gained, it becomes less necessary to think of variables as numbers. The thinking simply focuses on the letters (e.g., $2x+3x = 5x$ without thinking of $x$ as a number). Thus, the variables become the focus or the “object” of thought. At this point, the numbers and arithmetic have been abstracted to variables and algebra. Overall, what this means is that the development from the real world items to variables and algebra involves two steps:
(1) abstraction from items to numbers and arithmetic, and (2) abstraction from numbers and arithmetic to variables and algebra. That is, algebra is an abstraction of an abstraction (see Figure 8).

Figure 8. Algebra as an abstraction of an abstraction.

6.2 Abstraction, structure and holistic teaching

Recapping, the abstraction from arithmetic to algebra is an abstraction from particular activities represented by numbers and operations (i.e., arithmetic) to generalised activities represented by variables and operations (i.e., algebra). These generalised activities are interesting in that, to hold for all numbers, they must reflect structural things in arithmetic. In fact, they reflect what is called the underlying structure of arithmetic. This means, at its most powerful level, algebra reflects the “big ideas” in arithmetic – ideas that hold for whole numbers, fractions, measures as well as variables.

These big ideas are always present in what we do in arithmetic but are often undeveloped. A particular example may help in discussing this.

Example:

The new mental computation approaches to computation are recommending that addition tasks such as $25 + 48$ should be done by a strategy called compensation; that is $25+48$ is calculated by changing one of the numbers to something easy to add and then compensating for this change on the other number. Because 50 is easy to add, we could change 48 to 50 by adding 2 and compensate by changing 25 to 23 by subtracting 2. In this way, the addition can be easily calculated (i.e., $25+48 = 23+50 = 73$).

Most teachers stop here; they teach the strategy then support students to use it on other examples. To build big ideas, they need to go further. The important question that should be followed up is, “why does this work?”

The reason is that $23=25−2$ and $50=48+2$, so we are adding and subtracting 2. Since $−2$ and $+2$ are opposites or inverses, this is the same as adding 0, the identity (that which does not change anything). This means that the value of $25+48$ does not change when it becomes $23+50$ because all we are doing is adding 0. Putting in all the steps, what we have done is:

\[
\begin{align*}
\text{Start} & \quad 25+48 = 25+48+0 = 25+48−2+2 = 25−2+48+2 = 23+50 = 73 \\
\text{Finish} & \quad \\
\end{align*}
\]

However, the big idea behind compensation is more than the $−2+2$ in this example. The big idea is that a first thing always equals a second as long as all we do is add 0 or something equivalent to 0. Thus to work out something complicated, all we have to do is find something the same as zero which
changes it to something simple. This is an idea that can help us right across all mathematics (that is why it is called a big idea, or in mathematics terms, a principle). For example,

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>ZERO</th>
<th>WORKING</th>
</tr>
</thead>
<tbody>
<tr>
<td>238+387</td>
<td>−13+13</td>
<td>238+387 = 225+400 = 625</td>
</tr>
<tr>
<td>4.8+2.7+3.6</td>
<td>+0.2+0.3−0.5</td>
<td>4.8+2.7+3.6 = 5+3+3.1 = 11.1</td>
</tr>
<tr>
<td>3h48m+2h37m</td>
<td>+12m−12m</td>
<td>3h48m+2h37m = 4h+2h25m = 6h25m</td>
</tr>
<tr>
<td>2a^2−b^2</td>
<td>−ab+ab</td>
<td>a^2−ab+ab−b^2 = a(a−b)+b(a−b) = (a+b)(a−b)</td>
</tr>
</tbody>
</table>

This example shows a general method for teaching difficult additions by changing them to simple additions by finding things equivalent to zero to add to them. It is an example of algebra in action because the method does not identify the actual numbers to be used to bring about the change to a simpler form addition. Basically, the method says that any numbers will do as long as they add and subtract to 0. Thus, the example provides evidence for the power of big ideas, underlying mathematics structure, and teaching mathematics using algebra. Algebra teaching that is based on the structure of mathematics can enable students to learn big ideas that they can apply to particular examples right across mathematics.

In many schools, the teaching of arithmetic tends to focus on mathematics as disconnected parts, teaching the next activity as if it is a completely new thing, and relying on the weight of years for students to put together all the bits to make a whole. We will call this part to whole teaching. What Example 1.2 shows us is that algebra enables us to teach the more powerful mathematics where we learn a big idea and use it in particular situations. We shall call this whole to part teaching or holistic teaching.

Thus, algebra based on the structure of mathematics gives us a chance to teach holistically from the big picture down to the special case.

### 6.3 Indigenous culture, mathematics and holistic teaching

A danger in teaching Western mathematics (and science) to Aboriginal and Torres Strait Islander people is that teachers can make their teaching become a celebration of the growth and success of Western or European knowledge. It is particularly easy to represent Western knowledge as successful because it can be presented as continually advancing in terms of technology (e.g., cars, planes, rockets, computers) and as coming to dominate the planet. However, this same knowledge has been particularly unsuccessful in handling the intransigent and long-term problems of the planet such as destruction of the environment, poverty, war and violence, and global warming.

Teaching that presents mathematics as a celebration of this “linearly advancing technological process” can marginalise Aboriginal and Torres Strait Islander people, undermine the significance of their Indigenous identity and devalue Indigenous knowledges and cultures as simplistic societies (Matthews, 2003). Western mathematics places importance on number and arithmetic because this is where linear advancements in technological progress starts and what drives its progress. However, it can be argued that the invention of arithmetic was a consequence of a society in which material assets were considered more important than the individual. In Indigenous society, without the need to work out one’s assets in fine detail, number was not developed to the same level as in Western culture. However, this does not mean that Aboriginal and Torres Strait Islander cultures do not have their own mathematical knowledge.

Considerations of Indigenous knowledge of mathematics require recognition and respect of such knowledge, which should, in turn, be reflected in the teaching and learning of mathematics to
students. For example, as argued by Matthews (2003), Yolngu children, from a young age, have a
good understanding of their kinship system which governs the Yolngu way of life. This system is very
complex and relies on cyclical and recursive patterns. Such patterns can be found within numbers
themselves and other areas of mathematics (Jones, Kershaw & Sparrow, 1996; Divola & Wells, 1991)
and forms a good basis for Yolngu children to start their journey into Western mathematics. As most
Aboriginal and Torres Strait Islander knowledge systems are based on interactions within the
environment and groups of people, they can form algebraic systems because they can relate
numbers in flexible ways.

Traditionally, in Queensland schools, mathematics and its teaching both reflect Western culture.
Therefore, differences in mathematics performance can stem from a different cultural view of what
it means to be good at mathematics. Commonly, in most school environments, this is determined by
gauging students’ performance levels from test items that reflect non-Indigenous learning styles,
namely solving meaningless problems by pen-and-paper means. In those problems, there are often
marked differences in errors between Indigenous and non-Indigenous students. One case study of
Indigenous students’ errors found that underperformance tended to reflect mistakes in procedures
rather than understanding (reflecting the position of Grant (1998) that Indigenous students see the
whole rather than the parts).

Therefore, it is important to teach mathematics on an equitable basis with Western
mathematics reflecting “both ways” approaches (Ezeife, 2002). Western teaching is traditionally
compartmentalised, resulting in an education system in schools (whether oral or written) focusing on
the details of the individual parts rather than the whole and relationships within the whole. By
contrast, Indigenous students tend to be holistic learners, appreciating overviews of subjects and
conscious linking of ideas (Grant, 1998). In fact, Indigenous people have been characterised as
belonging to “high-context culture groups” (Ezeife, 2002) which are characterised by: a holistic (top-
down) approach to information processing in which meaning is “extracted” from the environment
and the situation. Low-context cultures use a linear, sequential building block (bottom-up) approach
to information processing in which meaning is constructed (Ezeife, 2002).

What this means is that students who use holistic thought-processing are more likely to be
disadvantaged in mainstream mathematics classrooms. This is because Westernised mathematics is
largely presented as hierarchical and broken into parts with minimal connections made between
concepts and with the children’s culture and community. It potentially conflicts with how they learn.
If this is to change, curriculum and assessment need to be made more culturally-sensitive and
community-orientated (see Deadly Maths Consortium, Tagai09 Maths for Employment Interim
Report, June 2009).

Thus, we have a confluence of results. Indigenous students are high context and learn best with
holistic teaching (Grant, 1998; Ezeife, 2002). Mathematics in its most powerful form is based on
structural understanding that is learnt best by holistic teaching. Algebra is the component of
mathematics that is based on mathematical structure, and is capable of presenting mathematics
holistically.

As a consequence, it seems that algebra is the form of mathematics that is most in harmony
with Indigenous culture and learning style. Because of this, algebra understanding should be a
strength of Indigenous students if it is taught through pattern and structure (rather than through
sequential teaching of rules and algorithms). It seems likely that algebra is a subject in which
Aboriginal and Torres Strait Islander students should excel. Finally, because of its relationship with
arithmetic, this understanding of algebra should enable enhanced understanding of and proficiency
with arithmetic.
6.4 Cultural implications for teaching Algebra

There are two implications for Algebra from the discussion above: (a) what is the best way to teach it, and (b) what is the best way to teach it to Aboriginal and Torres Strait Islander students?

1. Teaching Patterns and Algebra. The power of mathematics lies in the structured way it relates to everyday life. Knowledge of these structures gives learners the ability to apply mathematics to a wide range of issues and problems. This is best achieved if the knowledge is in its most generalised form, which is algebraic form. Thus, the most effective way to present mathematical knowledge is through algebra. However, any topic of mathematics can be presented instrumentally (as a set of rules). Although algebra is the direction for power in mathematics, it has to be algebra that is presented structurally, showing the generalisations that can be used in many examples. Powerful algebra teaching focuses on extending arithmetic to generalisations that can apply across all arithmetic. That is, teaching that builds holistic understandings of structure that can then be applied to particular instances (from the whole to the part). If students are fortunate enough to gain this structured understanding of mathematics, the subject becomes easy. This is because it is no longer seen as a never-ending collection of rules and procedures but rather the reapplication of a few big ideas.

2. Teaching Indigenous students. Aboriginal and Torres Strait Islander students tend to be high context. Their learning style is best met by teaching that presents mathematics structurally without the trappings of Western culture. Thus, powerful Indigenous teaching is therefore holistic, from the whole to the part. As Ezeife (2002) and Grant (1998) argue, Indigenous students should flourish in situations where teaching is holistic (from the whole to the parts). Thus, holistic algebra teaching has two positive outcomes for Indigenous students: (a) it teaches a powerful form of mathematics; and (b) it teaches it in a way that is in harmony with Indigenous learning styles. Algebra taught structurally then, is something in which Indigenous students should excel. However, this is just a general finding. What does this mean in practice for the teaching of algebra? First it means that we will not be teaching rules for manipulating letters. Letters and algebraic expressions and equations will be understood in terms of everyday life and algebraic ideas will be generalised from arithmetic. This will mean a lesser focus on algorithms and rules, and a greater focus on the following.

1. **Concepts.** These are meanings of mathematical ideas which apply across all arithmetic. For example, the meaning of a fraction as a “part of a whole”, where the whole is divided into equal parts, can be extended to percent, chance, probability, and rate and ratio.

2. **Strategies.** These are general “rules of thumb” that apply across all mathematics. For example, the separation strategy states that to add $48 + 27$ we separate into tens and ones, add separately and then combine. This strategy can be extended to apply to time, metric measures, mixed number and algebraic expressions. In algebra itself, the strategy is used to simplify expressions, (e.g. $2x+3y+4x+5y = 2x+4x+3y+5y = 6x+8y$).

3. **Principles.** These are relationships that hold regardless of context and thus apply across all mathematics. For example, “turn arounds” (the commutative principle) states that order does not matter in addition and multiplication. This rule can be applied usefully in number, measures and algebra itself (including functions and calculus).

Interestingly, holistic teaching is also positive for low SES students as well as Indigenous students, as low SES students tend to have strengths with intuitive-holistic and visual-spatial teaching approaches rather than verbal-logical approaches. Thus an algebraic focus on teaching mathematics should also be positive for low SES students.
7. Teaching Algebraic Structure

This section overviews the role Algebra plays in the structure of mathematics, describing how it is connected to the other strands within the structure of mathematics and how it is based on a series of big ideas or principles that recur across Years PP to 9. It is presented under the following headings: Connections and big ideas,

- **Structural connections** – arguing for the importance of connections and showing the place of Algebra within school mathematics; and
- **Big ideas/Principles** – overviewing how algebra sets up the big ideas and why this is important for teaching and learning of mathematics.

7.1 Connections and big ideas

YuMi Deadly Maths believes that mathematics should be taught so that it is accessible as well as available, that is, learnt as a rich schema containing knowledge of when as well as how (see Philosophy and Pedagogy booklet). Rich schema has knowledge as connected nodes which facilitates recall (it is easier to remember a structure than a collection of individual pieces of information) and problem solving (content that solves problems is usually peripheral, along a connection from the content on which the problem is based). As described in Table 1, the reality and mathematics components of the RAMR cycle are built, in part, around connections and generalisation to big ideas.

**Connections**

As a consequence, YuMi Deadly Maths argues that knowledge of the structure of mathematics, particularly of connections and big ideas, can assist teachers be effective and efficient in teaching mathematics. This is because it enables teachers to:

(a) **determine what mathematics is important to teach** (mathematics with many connections or based on big ideas is more important than mathematics with few connections or little use beyond the present);

(b) **link new mathematics ideas to existing known mathematics** (mathematics that is connected to other mathematics or based on the one big idea is easier to recall and provides options in problem solving);

(c) **choose effective instructional materials, models and strategies** (mathematics that is connected to other mathematics or based around a big idea commonly can be taught with similar materials, models and strategies); and

(d) **teach mathematics in a manner that makes it easier for later teachers to teach more advanced mathematics** (by preparing the linkages to other ideas and the foundations for the big ideas the later teacher will use).

Thus it is essential that teachers know the connections and big ideas so that they know mathematics that precedes and follows what they are teaching, because they are then able to build on the past and prepare for the future. Algebra is particularly important here.

The major structural connection for algebra is that it generalises arithmetic and so follows on and extends number and operations (see Figure 4). However, arithmetic also is part of
Measurement, Probability and Statistics, so Algebra also has connections here (particularly in Measurement where it relates to formulae). Thus, with input from Geometry, we have the connections of Figure 9.

![Figure 9. Connections between strands and within Number strand](image)

**Big ideas and algebra**

Big ideas or principles are mathematical ideas whose meaning is encapsulated in the relationship between the components of the idea not in the actual content focus of the idea. For example, the “turn around” strategy for basic number facts is based on the fact that a first number add a second number is the same as its reverse (the second number add the first number – e.g., 3+5=5+3). However, this idea also holds for large numbers (e.g., 3789+2094 = 2094+3718), fractions (e.g., 2/5+7/8 = 7/8+2/5), decimals (e.g., 4.39+6.68 = 6.68+4.39), algebra (e.g., a+b = b+a), and functions (e.g., f+g = g+f where f(y) = 2y-1 and g(y)=3y-2). Thus, as a mathematical idea, “turn around” is a big idea because it applies to any content and across the Years PP to 9. As such, it has a special name within the structure of mathematics – the commutative principle or law.

However, there are also other terms that can be applied to big ideas (some of this was discussed in the first section of this resource booklet). These can be seen in the example above. First, the commutative law can be said to generalise all the particular examples of turn arounds – so big ideas are generalisations. Second, the mathematical idea of commutativity can be seen as a large idea in which all the various turn arounds sit – so big ideas are holistic in their focus.

Because Algebra expresses generalities that hold for all numbers (and measures and probabilities, and other areas in which number and operations are used), then it can be argued that Algebra:

(a) is generalisation;
(b) consists of big ideas; and
(c) is holistic in focus.

As a consequence of what they are, big ideas/principles, generalities and holistic ideas: (a) last learners many years; (b) are available to connect many mathematical ideas; and (c) cover many mathematical situations; They reduce the amount of mathematics that has to be learnt and so they are a very powerful way to teach and learn mathematics. Thus, as it covers many generalisations and big ideas in a holistic manner, Algebra is also important and powerful in terms of teaching and learning mathematics.
Algebra focuses on some big ideas/principles more than others in the way it is used in PP-9 mathematics where it is connected to arithmetic. These big ideas/principles are:

(a) *global* – symbols tell stories, change vs relationship, and interpretation vs construction;
(b) *equals and order* – reflexivity, symmetry, transitivity, and balance;
(c) *operations (field)* – identity, inverse, commutativity, associativity, and distributivity; and
(d) *extension* – compensation, equivalence, backtracking, inverse relation, and triadic relationship.

All of these big ideas/principles have been previously described in the *Teaching Number* and *Teaching Operations* Resource booklets except for the change vs relationship big idea/principle, and the equals and order big ideas/principles which are described in this resource in the section called *Arithmetic principles*.

### 7.2 The power of algebraic big ideas

In this section we will show, in a larger example than that in section 2.1, how teaching a powerful algebraic principle can enable a lot of particular mathematics ideas to be achieved across many years of schooling. The example is the inverse principle, one of the most powerful principles or big ideas in mathematics, algebra and arithmetic.

**Models for teaching the inverse principle**

The inverse principle is the understanding that each operation has a partner operation that reverses or returns it to where it started. For example, +2 is reversed by −2 and ÷6 is reversed by ×6. As this is true for any number, measure or expression, therefore the operations themselves can be seen as inverses, that is + is the inverse of − and × is the inverse of ÷. Inverse, along with identity (the number that does not change anything, e.g., +0 and ×1) is one of the most important principles. It can be introduced in many methods; here we have three of them.

1. **Undoing.** In this method, the teacher begins by talking about actions and how they can be undone so you return to where you started. For example, turning to the right can be undone by turning to the left, and stepping forward can be undone by stepping back. The second step is for the teacher to act out a series of undoings: for example, step to right \(\rightarrow\) step to left, move hands clockwise \(\rightarrow\) move hands anti-clockwise, and so on. This notion of undoing can be used to introduce a lot of important inverses. [Note – the answers are in square brackets like this.]

   Take five counters and join four counters to them, to make nine. Discuss how we can undo this. [Take the four counters away and return to five]

   ![Diagram](image)

   This can be understood as +4 being undone by −4. Further examples will enable students to see that + is undone by −.
For two-step problems look at, for example, how putting on a sock and a shoe is undone:

Step 1 requires putting the sock on, and step 2 putting the shoes on.

Undoing requires taking the shoe off, and then taking the sock off, returning to a bare foot. We can see that each action is undone but also that both actions are undone in the opposite order (i.e., socks on, shoes on is undone by shoes off, socks off). Experiencing further examples leads to seeing that +3 +2 is undone by −2 −3.

2. **Function machines.** In this method, the teacher builds a function machine “robot” from a large box, hangs an operation around its neck, and develops sets of cards (e.g., input cards 1 to 20, output cards 1 to 30). A student hides inside the box, other students put in different number input cards and the hidden student puts out the appropriate number output cards on the opposite side of the box. The final (output) number cards are calculated by following the operation on the front of the robot (see diagram below).

![Diagram of function machine](image)

This is represented by “Arrowmath” notation as a change; e.g.,

\[ +2 \]

The question is asked, **What if you get 10 at the end? What was put in at the start?** Most children will be able to say 8 with support and most classes will be able to say that to “go backwards” requires subtracting 2 (after discussion and examples). In Arrowmath notation, this reversal of the operation can be represented as a change in the reverse direction (which inverts the operation), e.g.,

\[ +2 \xrightarrow{\text{Inverse}} -2 \]
Multiple operations can be handled by two function machine robots, e.g.,

This enables inverses of sequences of operations to be studied, e.g.,

\[
\begin{align*}
\text{Forward:} & \quad 5 \rightarrow 15 \rightarrow 13 \\
\text{Backward:} & \quad 5 \leftarrow 15 \leftarrow 13 \quad \text{(inverse)}
\end{align*}
\]

In practice, it is effective to actually walk students forward past the function machine from left to right and backward from right to left, verbalising each function as they walk past. This process assists students with grasping change and reversals. The construction of function machines can be novel and may appeal to students’ imaginations: using a large enough box so that a “wheel” can be attached, a student can turn the “wheel” to indicate something is happening, with another student sitting inside as the “machine” to indicate something is going to come out, thus showing change.

3. **Number line.** In this method, the teacher shows a number line (numbered) and discusses what happens as we move back and forth along the line. Says, *We are at 7, 2 is added, how do we get back to 7?*, and acts this out along the line showing inverse, for example:

Then, the teacher moves onto an unnumbered line. The teacher says, *Your dad gives you money and you spend $8, what has to happen to get back to the same amount of money you were given?* Students act this out on the number line with \(n\) as the letter signifying the money that Dad gave. Teacher discusses how to get back to \(n\). [Find someone to give you $8.] It shows that +8 is the inverse of −8.

If the number line is made “mathematical” so that all operations are possible, then we can use the line to do sequences of inverses.

(Note: for younger students inverses for + and − can be “walked” on a number track, e.g., 3 goes to 7 by +4 and then 7 goes back to 3 by −4, as on right.)
Applying the inverse principle

Once a learner has an understanding of inverse, it can be applied to particular mathematics topics. The important thing here is that this one piece of knowledge, the knowledge of inverse, because it is general and across topics, can be used to understand and solve problems in a number of different topics; separate rules or algorithms do not have to be taught for each topic because the inverse knowledge is enough. (Note: It is important to realise that the models used to teach the inverse principle also play an important role in the applications). Some examples of inverse applications are as follows.

1. **Basic subtraction facts.** If addition facts (e.g., 8+5=13) are known, inverse can be applied to calculate the subtraction facts (e.g., 13−8=5). This is based on subtraction being the inverse of addition and using this to rethink subtraction in terms of addition (and, therefore, using our addition facts to calculate the subtraction facts). For example, using function machines and Arrowmath notation, 13−8 can be thought of as:

   ![Function machine diagram]

   Where, reversing the notation, the inverse is:

   ![Arrowmath notation diagram]

   Similarly, on a number line (see below), 13−8 is as follows in terms of inverse.

   ![Number line diagram]

   From both of these models, 13−8=? Can be seen as the same as ?+8=13 (or “what plus 8 equals 13”). Thus, 13−8 can be calculated using the already known addition fact, 5+8=13. This means 13−8=5. Thus we have a strategy for solving subtraction facts which comes from the inverse big idea – it is called the “think addition” strategy.

2. **Subtraction computation.** Once you have learnt addition computation, inverse can be used in two ways in subtraction. The first is to check the subtraction as below.

   Subtraction: 52
   -27
   25
   Check by addition: 25
   +27
   52

   (This method can also be used to check additions, by using subtraction, multiplications by using division, and divisions, by using multiplication).

   The second way is as a method that uses addition to do subtraction computations. It is best seen with the number line model. To do the method, students have to “think addition” (similar to basic facts) and solve subtractions such as 52−27 by thinking “what has to be added to 27 to make 52”. This can be solved using an alternative jump method as shown below where start from 27 and determine what jumps will get to 52, as in the diagram below:

   ![Number line diagram]

   The answer is 3+10+10+2 = 25.

   This additive subtraction method is also useful for subtracting money (e.g., working out change), mixed numbers, decimal numbers, and measures (time and length).
3. **Solutions of linear equations.** Situations like *I bought $3 pies for all of my friends and a $7 roll. I spend $25. How many friends?*, can be considered in terms of change, can be acted out on ×3 and +7 function machines, and can be represented by Arrowmath notation and an equation, e.g.:

\[ \begin{align*}
? & \iff 25 \\
3x & \iff 7 \iff 25 \\
\end{align*} \]

The inverse or backtracking method that is part of learning inverse on function machines can be used to solve the problem. Since the change is ×3 and +7, it is reversed by −7 and ÷ 3. This reversing or backtracking provides the answer, for example:

\[ \begin{align*}
3x + 7 &= 25 \\
3x &= 25 - 7 = 18 \\
x &= 18 ÷ 3 = 6
\end{align*} \]

4. **Solving % problems.** Situations like, *I paid a 40% down payment of $120 on the dress, how much was the dress?* can also be thought of as change and solved by reversing, or finding the inverse of, the change. Both the function machine and number line models can assist here and the use of each is given as follows.

**Function machine or change model:** The problem can be considered as a function machine that changes the cost of a dress to 40% of that cost, that is, that multiplies original cost by 0.4. This can be represented by an Arrowmath diagram and solved by reversing or backtracking, as follows. The original cost of the dress is therefore found by dividing the 40% cost by 0.4, e.g.,

\[ \text{Cost of dress} \times 0.4 = 120 \iff \text{Cost of dress} = 120 ÷ 0.4 = 300 \]

**Double number line model:** The problem can also be considered as a change on a line. An excellent way to do this is to consider the line as having two sides (this is called a double number line) – one side as % and the other as $. In this situation, the 100% changes to 40% while the original cost changes to $120. The change on both sides is the same. From 100% to 40%, the change is to multiply by 0.4, so to go the other way, or undo the change, is to divide by 0.4. This means that the original cost (the ? in the diagram) is $120 ÷ 0.4 = $300.

\[ \iff \begin{align*}
\% & \iff 40\% \\
$ & \iff 120 \\
\end{align*} \]

5. **Solving rate problems.** For problems like, *I bought the petrol for $1.40 per litre, how much petrol did I buy for $63?,* both the function machine and number line models again apply.

**Function machine or change model:** Once again, the problem is considered as change, but change from litres to dollars by multiplying by the rate of change (i.e., ×1.40), e.g.,

\[ \text{Litres of petrol} \times 1.40 \iff \text{money in$} \]

Thus, we use the Arrowmath notation to set up the change and use the inverse (i.e., ÷1.4) to solve it, e.g.,

\[ \iff \begin{align*}
? \iff L \times 1.40 \\
\iff 63 \div 1.40
\end{align*} \]

By using inverse or backtracking in this way, the number of litres is \( = 63 ÷ 1.4 = 45L \).
Double number line model: Once again, we can use the double number line with one side \(L\) (litres) and the other side \(\$\) (dollars). In this situation, \(1\)L is \(\$1.40\) so the line becomes as below. To get from \(\$1.40\) to \(\$63\) is to multiply by \(63\) and divide by \(1.4\). Thus, the number of litres of fuel is \(? = 1 \times 63 \div 1.4 = 45\)L.

Overall, learning the inverse principle enables you to solve many problems in many areas of mathematics. The big idea, inverse, enables a whole collection of what is often seen as distinct mathematics situations to all be solved with the one idea.

[Note: It should also be noted that the 5 applications above are only a few of the uses of inverse. It is also useful for ratio, measurement (metric conversion), currency conversion, and scale problems.]

7.3 RAMR Cycle

As described in Resource 1, Philosophy and Pedagogy, and encapsulated in Figure 10, the YuMi Deadly Maths Program takes the view that mathematics is:

- a creative, symbolically-based and culturally-biased representation of reality;
- a human construction (historical and social) through abstraction and reflection;
- an interconnected structure of ideas, a concise language and a collection of tools for solving problems; and
- a powerful generic thinking framework that can be applied to all parts of life.

![Figure 10. Relationship between perceived reality and created mathematics (Matthews, 2006).](image)

This Figure sees mathematics is an abstraction from the real world which results in specialised ways of describing and relating that are based on symbols. As argued in the Philosophy and Pedagogy resource, this abstraction and the necessary reflection back into the real world is highly creative but is also affected by the cultural bias of the abstractor and reflector. As school mathematics reflects the dominant middle class culture of mainstream Australia, this means that care must be taken in teaching to ensure that this bias is not present in instruction to Aboriginal, Torres Strait Islander and low SES students whose learning styles are better met through holistic teaching than the step-by-step teaching approaches used in traditional mathematics instruction. Interestingly, holistic whole-to-part teaching is very effective instructional approach for mathematics and, in particular, Algebra; more effective than the part-to-whole approaches used in traditional mathematics teaching.
RAMR framework

As also described in *Philosophy and Pedagogy*, Figure 10 is the basis of a Pedagogical Framework for teaching mathematics to Aboriginal, Torres Strait Islander and low SES students which:

(a) takes account of cultural differences between students and mainstream mathematics teaching;
(b) integrates mathematics teaching with school change and leadership processes that build pride, positive identity and high expectations (see School Change and Leadership resource);
(c) deconstructs Figure 10 to a reality-abstractation-mathematics-reflection (RAMR) framework from which sequences of instruction can be developed (see Figure 11).

Figure 11 describes this framework. It begins with the real world and abstracts real situations through modelling to formal language and symbols. It builds mathematics as a structure, and then it extends it by reflection on flexibility, generalisation and reversing.

Using the RAMR framework, student learning activities are based in reality, the real world or students’ real life experiences. This provides a good base for building mathematical understanding in the context in which it appears. From there the cycle works through Abstracting to Mathematics and then uses Critical reflection to return the abstracted mathematics back to its application in Reality (or context).

Abstracting is the process used to gradually reduce the level of contextual information from the real world context (reality) until only the abstract symbolic (or formal) mathematics exists that has been generalised and can be applied back into any context. To scaffold students through the abstracting process, materials and models are used in a sequence that travels from high context (real world items) to generalised or no context (symbols). The following sequences and models assist with this process and should become part of the natural order of planning abstracting activities.

---

**Figure 11. RAMR framework**

- **Reality**
  - Identify local cultural and environmental knowledge that can be used to introduce the idea.
  - Ensure existing knowledge prerequisite to the idea is known.
  - Construct kinaesthetic activities that introduce the idea (and are relevant in terms of local experience).

- **Abstraction**
  - Develop a sequence of representational activities that develop meaning for the mathematical idea.
  - Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
  - Allow opportunities to create own representations, including language and symbols.

- **Reflection**
  - Set problems that apply the idea back to reality.
  - Lead discussion of idea in terms of reality to enable students to validate and justify their own knowledge.
  - Organise activities so that students can extend the idea (use reflective strategies – being flexible, general-issing, reversing, and changing parameters).

- **Mathematics**
  - Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
  - Facilitate students’ practice to become familiar with all aspects of the idea.
  - Construct activities to connect the idea to other mathematical ideas.
7.4 Sequences and models to scaffold abstraction

Body→Hand→Mind

**Body**: Students need to build strong mental models for mathematics learning to be robust. Sequences of abstracting should start with bodily kinaesthetic activity that is clearly linked to the concept being built.

**Hand**: These activities should be directly followed by modelling activities using materials (or manipulatives) that can be moved by hand to represent the real life context. Students should be encouraged to articulate their thinking and understanding in their own language initially, but also assisted to link to further vocabulary for the concept that is linked to everyday language, mathematical language and symbolic language. Working through the language sequence builds very closely onto mind activities.

**Mind**: Once students have completed activities with manipulatives it is possible to link to language and symbolic representations towards abstract mathematics. Students can be encouraged to symbolise the mathematics they have experienced using their own symbols to create ownership and deeper understanding of the meaning of symbols (see MAST in Philosophy & Pedagogy).

Once students understand the working of symbols it is possible to follow very closely with the need for shared understanding of symbols and standard mathematical symbols in use. This leads naturally into the Mathematics phase of the RAMR cycle where students are able to work with symbolic mathematics separate from its original context. This generalisation should be actively discussed and applied to other contexts during the Critical Reflection phase of the cycle.

**Representations**

Working with concrete materials to represent real world contexts need to also gradually decrease in the amount of contextual information they provide for students as shown in Figure 12.

![Figure 12. Sequence of concrete material representations.](image)

When using models for counting it is important to remember to include both discrete and continuous models to assist students with understanding the Discrete vs Continuous big idea as well as providing enough examples of how unit appears in the real world to ensure students build flexible and robust understandings. The following models need to be used in number experiences to ensure that student knowledge can be applied flexibly in real world situations. Note that Continuous models from the real world become discrete and countable when arbitrary measures are applied. For example, water in a jug with no markings is continuous; marking increments up the side breaks the continuous into discrete countable units. Similarly a streamer is continuous but its length becomes discrete and countable when sat along a ruler or measuring tape.

![Figure 13. Discrete vs Continuous.](image)
Some real world models for discrete, continuous, continuous made countable (or discrete) are pictured below showing set model, volume model, length model and area model.

Abstraction with materials progresses in a sequence from concrete to symbols as in Figure 14.

Concrete

- Real-world (e.g., biscuits) and replicas (e.g., cars)
- Manipulatives (e.g., counters, Unifix, bundling straws, MAB)
- Virtual (computer replications of materials)
- Pictures (of the above, diagrams, maps, number lines)
- Symbol patterners (e.g., calculators, spreadsheets)

Abstract

*Figure 14. Sequence of working from concrete materials to abstract symbols.*

Models and representations, language and symbols need to be effectively connected so that students can move back and forth flexibly between the three when interpreting and solving mathematical problems. Language progresses through a sequence and children need to become familiar with the various mathematical meanings inherent in language in order to apply their mathematics effectively to real world situations. Language should be built in complexity according to the following sequence:

Child language → Everyday language → Materials language → Symbols

The Payne and Rathmell (1977) triangle (Figure 16 in Philosophy and Pedagogy resource, here Figure 15) is an excellent vehicle for connecting real world problems, representations, language and symbols. Activities and questions should be constructed that encourage students to connect and move flexibly between model, language and symbols in all directions.

*Figure 15. The Payne and Rathmell (1977) triangle for early number.*
The YDM framework is designed to enrich the ACARA curriculum. This bolstering does not result in new work, just a better way of developing teaching that reflects big ideas and is more in harmony with Aboriginal, Torres Strait Islander and low SES students.

8.1 Justifying the framework

YuMi Deadly Maths is based on connections and big ideas. For arithmetic (number and operations), these come from algebra and lead to algebra. The best way to learn mathematics is by using these connections and big ideas – that is, using algebraic thinking to generalise arithmetic. This needs to be developed early so that students can better understand number and operations and be better prepared for formal algebra. It results in a lot of interesting activities that assist necessary number and operations work to be learnt but also pre-empt later algebra work. As argued in the previous section of this resource, this leads to a more effective and powerful way to teach mathematics. As discussed in the first section of this resource, the relationship between everyday life and algebra is a two-step abstraction that goes through arithmetic:

What consequences does this have for students trying to understand algebra? First, it means that the act of generalising is at the core of algebra and an understanding of: (a) what has to be generalised (the basic functions of mathematics); and (b) proficiency in both the act of generalising (how to generalise) and the products of generalisation (the mathematical ideas that result from generalising) must be built. Second, it means that the symbols of algebra, notably the letters, are far removed from everyday life and their meaning must be built with care through: (a) continuous connections being made between symbols and real-world stories; and (b) using sequences of materials and activities that become progressively more abstract. Thus, to work out what should be in an “algebraifying” of arithmetic, we look at:

(a) general approaches – generalising and generalisation, and change and relationship;
(b) focus and sequencing for early algebra – Maths as Story Telling and teaching sequences;
(c) final content framework based on the above; and
(d) yearly teaching frameworks PP-3 that detail how the above will be taught.

It is important also to ensure students understand that generalising requires a process (how to generalise) and gives a product (generalisations), and that mathematical stories can have two meanings, namely, relating and changing. These four become the content of algebra in PP-3.
Generalising and generalisation

To understand algebra, students will have to be taught to generalise and understand common generalisations. Generalisation is the act of grasping a pattern from particular examples. One excellent way to teach it is to find the pattern rule for patterns; that is, to consider sequences of terms, both visual and numerical, and to determine the rule that determines the 10th term, the 100th term, the 256th term and the general term, namely, the nth term. [Note: There are two types of pattern, repeating and growing. Many curriculums devalue the repeating pattern. However, as will be seen in this resource, repeating pattern activities enable powerful generalisations to be found.]

It is important to understand two aspects of generalisation.

1. There are also two parts to generalisation (as there are two parts to finding a pattern in patterning or another mathematical activity): (a) finding and determining the generalisation; and (b) expressing the generalisation.

2. There are four stages to building students’ ability to express a generalisation (or pattern or some other general rule): (a) students express how the rule will work for examples close to those that have been given (often using gestures to express themselves); (b) students express the rule for any number (including large numbers) – this is called quasi-generalisation; (c) students express the rule generally in language; and (d) students express the rule using variables (e.g., n).

Thus patterning has to be part of an early algebra curriculum. It teaches the act of generalisation, and is an excellent way to introduce the notion of variable.

Of course, once you have generalisation you end up with generalisations. Many of the generalisations from patterns are particular to the problems; but if the technique is applied to arithmetic equations, then the patterns that are found are often rules that hold for any number, measure and algebraic expression. Thus arithmetic principles are also a part of an early algebra curriculum.

Relationship versus change

Let us consider three examples:

1. The first is potatoes being cooked into chips. We can consider this as a relationship, the potatoes and chips are the same food; we can consider this as a change, the potatoes have been changed to chips by cooking. This gives rise to two different ways of thinking about and two different symbols for one mathematical idea – same as or equals, and change or arrow, as below.
2. The second is a balloon being blown up. The half filled balloon and the fully filled balloon can be considered to be related because they are the same shape although different in size. However, from another perspective, the little balloon could be considered to have been changed into the large by being blown into. Again we have two different symbols and two different ways of thinking (see below) for one mathematical idea.

3. **The third is Addition.** Consider 2 and 3. The joining of 3 to 2 could be considered as a relationship, “2 and 3 relate to 5 by addition”. However, it can also be considered as a change, “2 can be changed to 5 by the action of +3”. Again, one mathematical idea but two ways of thinking and two different forms of symbols, as seen below.

\[
2 + 3 = 5 \\
2 \xrightarrow{+3} 5
\]

So, the everyday activities such as addition can be thought of in two ways and result in two symbol systems even though they only represent one mathematical idea. One way uses equations to express relationships such as that between 2, 3, 5 and addition; the other way uses arrows to show change such as how 2 can be changed to 5 by addition of 3. The first way leads to algebraic equations; the second to algebraic functions.

This two-way method of looking at the world and mathematics holds for algebra. For example, when we say \( y = 2x + 1 \), we can be expressing an equation such as the value of the coat \( y \) was $1 more than double the value of the pants \( x \) or we could be expressing a function such as saying that to work out the price of the coat \( y \) we double the price of the pants \( x \) and add a dollar.

Thus, change (functions) and relationship (equations) has to be part of an early algebra curriculum. And, overall, YuMi Deadly Algebra has four strands: *Repeating and Growing Patterns*, *Arithmetic Principles*, *Equivalence and Equations* (note the use of equivalence instead of equals); and *Change and Functions*.

**Mathematics as story telling (MaST)**

As well as the above, early algebra has to focus on giving meaning to abstract symbols such as \( 2y+3 \). This means a heavy focus on symbols \( \rightarrow \) reality and reality \( \rightarrow \) symbols.

We will take first the symbols consequence. Most students cannot see the relevance of, say, \( x+y=7 \) to their everyday world. Yet, with understanding it is very relevant. It could mean that you bought two things at a shop for $7. Then the cost of the first thing \( x \) plus the cost of the second thing \( y \) is equal to $7. This gives parameters in which thinking can be used. Suppose we were working in whole dollars. Then the first thing could cost $1 and the second cost $6, or $2 and $5, or $3 and $4, and so on. Thus we need to teach students the role of symbols in telling stories. This is the basis of the MAST (Mathematics as Story Telling) activities developed in the Algebra project at Dunwich State School led by Chris Matthews described in the *Philosophy and Pedagogy* resource. It has 7 steps as the example below shows.
1. Symbols. Students explore how symbols can be assembled to tell a story, first in Indigenous situations (e.g., Indigenous art) and then creating and interpreting symbols for simple actions (e.g., walking and sitting at a desk).

2. Exploring simple addition. Students act out a story (e.g., 2 students join 3 students to make 5 people). Discussion identifies story elements – objects (the 2, 3 & 5 people) and actions (joining, making).

3. Creating own symbols. Students create their own symbols to tell the story. They first do this free style (a drawing that represents the “joining” and the “making”) and discuss results (e.g., are they linear – showing the action left to right, or are they more holistic)? Secondly, the students create symbols in a more structured and linear setup (students use objects for students and drawing for “join” and “make” or “same as”), as in the example on right.

Here the “join” picture is a vortex that picks up the 2+3 and the “making” picture is a cloud that brings them down together as rain.

4. Symbol showing. Students share symbols and explain their symbols’ meanings. Students then use other students’ symbol systems to represent other stories (e.g., 4 people join 7 people), and make up stories where other students’ symbols are used to represent addition.

5. Story modification. Teacher removes one counter from the left hand side 2 counters and asks if story still true (see example on right). Most students will say no. Teacher then asks how it can be made true again. The normal answers are:

   (a) put the counter back (as on right)

   (b) Add a counter to the 3 counters as on right (the compensation principle),

   (c) draw another vortex on the left hand side and put a counter in front of it.

6. Unknown. Teacher sets up story: “unknown number of people joined 3 to make 5”. Students create their own symbol for unknown as on right. Students use balance principle to find unknown.

7. Formal symbols. Teacher introduces students to common formal symbols for join (+), results (=) and unknown (x) e.g., 2+3 = 5 and x+3 = 5. Begin to relate these symbols to everyday situation and to solve for unknown.

Through these MaST activities, symbols are introduced as a shorthand language for telling stories. As well as this, the principles of balance and compensation are also introduced.
### 8.2 Yearly teaching frameworks

This framework has been modified to take into account the changes in the ACARA Mathematics Curriculum.

<table>
<thead>
<tr>
<th>SUB-STRAND</th>
<th>YEAR PP</th>
</tr>
</thead>
</table>
| **Repeating & growing patterns** | *Repeating patterns:* Sorting and classifying and explaining reasons; Following and creating sound and movement patterns; Copying and creating repeating patterns – one attribute repeats  
*Growing patterns:* Following patterns where one component grows in a counting manner. |
| **Arithmetic principles**      | *Equivalence and order principles:* Experience identical things having same length or balancing; Experience things remain in balance when order is reversed; Experience what happens when unbalanced is reversed in order.  
*Field and extension field principles:* Experience no change and changing and undoing change (e.g., turn 360, do nothing, one step forward and one step back, and so on) |
| **Equivalence & equations**    | *Meaning:* Introduce same and different – Sort and classify; Use balance and length to explore when things are same and different  
*Balance rule:* |
| **Change and functions**       | *Meanings and notation:* Explore idea of change (e.g., look at how walking two steps changes 3 steps to 5 steps, a drawing to a painted drawing, cooking, and so on); Set up simple function machine for changing colours, or language  
*Backtracking:* Explore getting input from output for simple example (“What did I start with?”). |
<table>
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<tr>
<th>SUB-STRAND</th>
<th>YEAR P</th>
<th>YEAR 1</th>
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</table>
| Repeating & growing patterns  | *Repeating patterns*: Copy, continue, create and describe repeating patterns with objects using more than one attribute.  
*Growing patterns*: Explore extending repeating patterns put in form of repeats to simple growing patterns.  
*Patterns in other strands*: Use numbers in repeating patterns, look at patterns in even/odd numbers. | *Repeating patterns*: Copy, continue, create and describe patterns with objects and numbers to 100 (e.g., RBGGRBGGGBGGG ..., 20 30 60 20 30 60 ...)  
*Growing patterns*: Extending repeating patterns to simple growing patterns with one attribute (e.g., RB RBB RBBB ...). Look at growing patterns with objects and numbers to 100 (e.g., 2 4 6 ..., 1 4 7 ...)  
*Patterns in other strands*: Explore place-value patterns (e.g., all numbers in forties have a 4 followed by a number, counting in tens from 6 always has a 6 in ones position). Explore counting patterns (e.g., odometer principle for ones and tend, patterns in counting by 2s, 5s, 3s, etc.) |
| Arithmetic principles         | *Equivalence and order principles*: Use balance and length situations to show reflexivity and symmetry. Relate these to reality and extend to counting and set and number line situations.  
*Field and extension field principles*: Use balances, number lines function machine situations to introduce identity (no change) and inverse (undo change). Relate inverse to joining and separating. Explore the MAST activity – allowing students to create their own operation symbols | *Equivalence and order principles*: Use balance and length situations to explore all equivalence and order principles (unnumbered → numbered). Relate to set models and number situations.  
*Field and extension field principles*: Continue with MAST activity. Use balance, function machine and line models to explore identity, inverse, and inverse relation. Relate to addition and subtraction (e.g., add 0, +3 and -3, increasing parts being joined increases answer but increasing take away part decreases answer). |
| Equivalence & equations       | *Meanings*: Relate same and different to equals and unequals. Use unnumbered contexts (e.g., weights and length) to explore equal and order relations; Introduce idea of equation/inequation to represent balanced and unbalanced situations.  
*Balance rule*: Discuss what happens to equals when weights are added or removed from sides of the balance, or lengths are added and cut off. *Relation to MAST activity* | *Meanings*: Use balance and line models in unnumbered situations to relate real world situations to equations and vice versa. Express equations with more than one object on either side. Move to numbered situations and express equations; Relate to real world situations; Show 5=2+3 and 6-1=2+3 are correct equations  
*Balance rule*: Look at adding and removing from either side; Express balance rule for unnumbered situations. Relate to MAST activity. |
| Change and functions          | *Meanings and notation*: Explore function machine for unnumbered situations. Look at two machines for two step changes  
*Backtracking*: Look at Output → Input as well as Input → output; Relate to real world situations Discuss what backtracking in two step unnumbered situations. Recognise there are two steps back as well as forward. | *Meanings and notation*: Explore function machine for unnumbered and simple numbered situations (addn & subtn); Relate to real world; Record Input & Output; Introduce Arrowmath notation  
*Backtracking*: Continue Output → Input activities for unnumbered and numbered situations; Discuss relation forward to backward change for addition and subtraction changes. Introduce backward Arrowmath. |
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<tr>
<th>SUB-STRAND</th>
<th>YEAR 2</th>
<th>YEAR 3</th>
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<tbody>
<tr>
<td>Repeating &amp;</td>
<td><em>Repeating patterns:</em> Complete repeating patterns with objects and numbers using more than one attribute.</td>
<td><em>Repeating patterns:</em> Copy, continue, create and describe patterns with objects and numbers to 100</td>
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<tr>
<td>growing patterns</td>
<td><em>Growing patterns:</em> Copy, continue, complete, create and describe simple growing patterns based on visuals and numbers; Explore sequential and position rules.</td>
<td><em>Growing patterns:</em> Explore extending repeating patterns put in form of repeats to simple growing patterns.</td>
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<td><em>Patterns in other strands:</em> Describe and predict place-value and counting patterns (e.g., skip counting, multn/constant addn patterns from 99 board, odometer rule)</td>
<td><em>Patterns in other strands:</em> Continue place-value and counting patterns for whole numbers and simple fractions, describing patterns (i.e., identifying next elements)</td>
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<td>Arithmetic principles</td>
<td><em>Equivalence and order principles:</em> Express equivalence and order principles in numbered situations; Relate to real world.</td>
<td><em>Equivalence and order principles:</em> Consolidate principles using language in numbered situation. Encourage to express as generalisations in equation form</td>
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<td><em>Field and extension field principles:</em> Explore compensation – use kinaesthetic length and mass situations (e.g., relay race). Explore identity, inverse, compensation, inverse relation and turn arounds (commutative) in numbered situations. Express as equations. Relate compensation to near 10 strategies and mental computation</td>
<td><em>Field and extension field principles:</em> Discuss using identity, inverse, compensation, inverse relation and turn arounds in real world situations – relate to arithmetic. Introduce associativity and equivalence. Relate to arithmetic.</td>
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<td>equations</td>
<td><em>Balance rule:</em> Introduce balance rule for numbered situations</td>
<td><em>Balance rule:</em> Encourage students to generalise balance rule for adding and subtracting. Use with physical and pictorial models.</td>
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<td><em>Unknowns:</em> Introduce notion of variable and relate to equations (let students choose their notation for unknown; Explore balance stories with unknowns); Focus on quasi-experimental and language descriptions; Discuss using balance rule with unknowns</td>
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<td>Change and</td>
<td><em>Meanings and notation:</em> Relate real world to machine, Input-Output and Arrowmath notation. Consolidate function machine work with addn &amp; subtn real world situations. Undertake simple arithmetical excursions – discuss what they say about the operations. Undertake Arrowmath activities with calculators.</td>
<td><em>Meanings and notation:</em> Introduce multn and divn on function machines. Continue arithmetical excursions. Check students can interpret world in terms of change. Discuss two step function machines.</td>
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<td>functions</td>
<td><em>Backtracking:</em> Relate backtracking to inverse of operations (addn/subtn). Discuss and relate to real world. Continue with Arrowmath</td>
<td><em>Backtracking:</em> Look at how backtracking will help find an unknown; Relate to inverse of operations (multn/divn); Look at backtracking when two steps.</td>
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<td><em>Solutions:</em> Introduce unknown by considering what happens Input is unknown – how can we think about the Output (focus on quasi experimental and language descriptions). Explore unknown stories (e.g. I have a number ...? and Jack and Fred both gave me the same amount of money, I spent ...?). Use number lines to follow and solve stories; Look at backtracking from an unknown output;</td>
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</tbody>
</table>

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References


